

SYMMETRIC STRUCTURE FOR THE ENDOMORPHISM ALGEBRA OF PROJECTIVE-INJECTIVE MODULE IN PARABOLIC CATEGORY

JUN HU AND NGAU LAM

ABSTRACT. We show that for any dominant integral weight λ of a complex semisimple Lie algebra \mathfrak{g} , the endomorphism algebra of any projective-injective module in the parabolic BGG category $\mathcal{O}_\lambda^{\mathfrak{p}}$ is a symmetric algebra (as conjectured by Khovanov) extending the results of Mazorchuk and Stroppel for the regular dominant integral weight. Moreover, the endomorphism algebra of any projective-injective module in $\mathcal{O}_\lambda^{\mathfrak{p}}$ equips with a homogeneous non-degenerate symmetrizing form.

1. INTRODUCTION

Symmetric algebras is an important class of algebras enjoying many good properties. The purpose of this paper is to show that the endomorphism algebra of any finite sum of indecomposable projective-injective (i.e. at the same time projective and injective) modules of integral weights in a fixed parabolic Bernstein-Gelfand-Gelfand (BGG) category is a symmetric algebra. In the first part of paper we study whether the endomorphism algebra B of the basic projective-injective module over any finite-dimensional algebra A is a symmetric algebra in general settings. The algebra B can be endowed with a non-degenerate associative bilinear form $(-, -)_{\text{tr}}$ which makes it into a Frobenius algebra, see in Proposition 2.11 below for precise statement. Furthermore, every indecomposable projective-injective module having an isomorphic head and socle is a necessary condition for the algebra B to be a symmetric algebra, see Lemma 2.4 and 2.12 below for the precise statements.

Now we assume that the head and the socle of every indecomposable projective-injective module are isomorphic and A is positively graded. The definition of $(-, -)_{\text{tr}}$ mentioned above relies on a prefixed basis of the algebra in question. A priori, it is unclear whether the non-degenerate associative bilinear form $(-, -)_{\text{tr}}$ is symmetric or not. In order to characterize the algebra B being a symmetric algebra, we propose a notion, called “admissible conditions”, on the homogenous bases of B . Roughly speaking, the admissible conditions are certain symmetric property of the multiplication for the homogenous basis of the algebra B . Under certain circumstance in the \mathbb{Z} -graded setting, we show that B is a symmetric algebra if and only if there exists an admissible basis of B , see Corollary 3.12. In this case, the form $(-, -)_{\text{tr}}$ is symmetric, see Proposition 3.9. Therefore every indecomposable projective-injective module has the same graded length in each block of B is a necessary condition for the algebra being a symmetric algebra following from the definition of admissible conditions. We show in Proposition 3.22 that there are some interesting classes of algebras such that every indecomposable projective-injective module has the same graded length in each block of the algebra.

Let \mathfrak{g} be a complex semisimple Lie algebra with a fixed Borel subalgebra \mathfrak{b} containing the Cartan subalgebra \mathfrak{h} and \mathfrak{p} a parabolic subalgebra containing the fixed Borel subalgebra \mathfrak{b} . For a dominant integral weight λ , let $\mathcal{O}_\lambda^{\mathfrak{p}}$ denote the subcategory of the parabolic Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}^{\mathfrak{p}}$ [18] with respect to the parabolic subalgebra \mathfrak{p} corresponding to λ .

The self-dual projective modules in parabolic BGG categories have been studied intensively in [10, 11, 12]. Note that the notion of the self-dual projective module is the same as projective-injective module (see Lemma 4.5 below). In those works, Irving attempted to speculate that every self-dual indecomposable projective module in $\mathcal{O}_\lambda^{\mathfrak{p}}$ always has the same Loewy length and hence has the same graded length. This speculation was proved by Mazorchuk and Stroppel [17, Theorem 5.2, Remark 5.3] in the cases when λ is any regular dominant integral weight and by Coulembier and V. Mazorchuk [15] in all cases.

1.1. Conjecture. (Khovanov) *For a dominant integral weight λ , the endomorphism algebra of any projective-injective module in $\mathcal{O}_\lambda^{\mathfrak{p}}$ is a symmetric algebra.*

2010 *Mathematics Subject Classification.* 17B10; 20G05.

Key words and phrases. Parabolic BGG category, socular weights, projective-injective modules.

The Conjecture 1.1 was proposed by Khovanov formulated in the beginning of Section 5 of [17]. Mazorchuk and Stroppel [17] proved the conjecture for the basic projective-injective module in \mathcal{O}_λ^p in the case when λ is a regular weight [17, Theorem 4.6] and hence the endomorphism algebra of any projective-injective module in \mathcal{O}_λ^p is a symmetric algebra. The purpose of this paper is to give a proof of the above conjecture for the remaining case, i.e., for any singular integral dominant weight λ . The main result of this paper is the following theorem.

1.2. Theorem. *For a dominant integral weight λ , the endomorphism algebra of any projective-injective module in \mathcal{O}_λ^p is a symmetric algebra.*

We sketch an outline of our proof. When working with the parabolic category \mathcal{O}_0^p , Mazorchuk and Stroppel's result [17, Theorem 4.6] ensures that there exists a symmetrizing form on the endomorphism algebra of the basic projective-injective module in \mathcal{O}_0^p . By the general theory developed in Section 3 described above, there exists an admissible basis for the endomorphism algebra of the basic projective-injective module in \mathcal{O}_0^p and the canonical form tr attached to it is a symmetrizing form. Now we apply the graded translation functors to connect \mathcal{O}_0^p with \mathcal{O}_λ^p , where λ is a singular dominant integral weight. Then we can use the crucial endomorphism $\bar{\theta}$ to define a form tr_λ on the endomorphism algebra of the basic projective-injective module in \mathcal{O}_λ^p through the graded translation functor and the canonical form tr . As a result, we can show with help of the technical Proposition 4.25 that tr_λ is a homogeneous symmetrizing form.

The paper is organized as follows. In Section 2 we work in a general setting to study the endomorphism algebra B of the basic projective-injective module over any finite-dimensional algebra. We first show in Proposition 2.11 that B with an anti-involution is always endowed with a non-degenerate form “tr” (equivalently, a non-degenerate associative bilinear form $(-, -)_{\text{tr}}$). In particular, B is a Frobenius algebra. The canonical form “tr” depends on the choice of a prefixed appropriate basis (Definition 2.6) of B . In Section 3 we work in a \mathbb{Z} -graded setting and propose a notion of admissible basis in Definition 3.7. We show in Proposition 3.9 that the canonical form attached to an admissible basis is always symmetric. In Corollary 3.12 we show that under certain circumstance in a \mathbb{Z} -graded setting B is symmetric if and only if there is an admissible basis for B . In Proposition 3.22 we show that with certain special assumptions in the \mathbb{Z} -graded setting (which are satisfied in the parabolic BGG category case too), all the indecomposable projective-injective modules in any fixed block of certain finite-dimensional algebra always have the same graded length. In Section 4 we give the proof of Theorem 1.2. The whole section is devoted to develop some properties on graded translation functors to show the existence of the endomorphism $\bar{\theta}$ and to prove the useful property of $\bar{\theta}$ described in Proposition 4.25 which is the main ingredient of the proof of Theorem 1.2.

Notations: We let \mathbb{Z} , \mathbb{Z}_+ and $\mathbb{Z}_{>0}$ denote the sets of all, non-negative and positive integers, respectively. Let \mathbb{C} denote the field of complex numbers. The identity map of a set X is denoted by id_X . The image of a function f is denoted by $\text{Im}(f)$.

2. ENDOMORPHISM ALGEBRAS OF PROJECTIVE-INJECTIVE MODULES

The purpose of this section is to work in a general setting to study the endomorphism algebra of the basic projective-injective module over any finite-dimensional algebra.

Let K be an algebraically closed field and let A be a finite-dimensional unital K -algebra. Let $A\text{-mod}$ denote the category of finite-dimensional A -modules. Let $\{L^\lambda | \lambda \in \Lambda\}$ denote a complete set of pairwise non-isomorphic simple A -modules, where Λ is a finite index set. From our assumption that K is algebraically closed, we have that

$$(2.1) \quad \text{End}_A(L^\lambda) = K \quad \text{for any } \lambda \in \Lambda.$$

For each $\lambda \in \Lambda$, let P^λ and I^λ denote the projective cover and the injective envelop of L^λ , respectively. For $M \in A\text{-mod}$, the socle of M , denoted by $\text{soc}(M)$, is the largest semisimple submodule of M and the radical of M , denoted by $\text{rad}(M)$, is the smallest submodule such that $M/\text{rad}(M)$ is semisimple. We will call $M/\text{rad}(M)$, denoted by $\text{head}(M)$, the head of M .

We let

$$\Lambda_0 := \{\lambda \in \Lambda | P^\lambda \text{ is an injective module in } A\text{-mod}\}.$$

Therefore Λ_0 parameterizes all the indecomposable projective-injective modules in $A\text{-mod}$. Henceforth, we assume that $\Lambda_0 \neq \emptyset$. For every $\lambda \in \Lambda_0$, there is a unique $\lambda' \in \Lambda$ such that $P^\lambda \cong I^{\lambda'}$ and $\text{soc}(P^\lambda) \cong L^{\lambda'}$ since P^λ is injective. It is clear that the map $'$ from Λ_0 to the set Λ is injective.

2.2. Definition. The module $\bigoplus_{\lambda \in \Lambda_0} P^\lambda$ is called the basic projective-injective module. We define

$$(2.3) \quad B := \text{End}_A\left(\bigoplus_{\lambda \in \Lambda_0} P^\lambda\right) = \bigoplus_{\lambda, \mu \in \Lambda_0} \text{Hom}_A(P^\lambda, P^\mu).$$

We assume that the K -algebra A equips an K -linear anti-involution $*$. For each $M \in A\text{-mod}$, we define the dual M^* of M as follows: $M^* := \text{Hom}_K(M, K)$ as a K -vector space, and $(af)(m) := f(a^*m)$ for any $a \in A$, $m \in M$ and $f \in M^*$. Then the dual $*$ is a contravariant exact functor from $A\text{-mod}$ to itself. It is clear that the functor $*$ gives an equivalence of categories since $(M^*)^* \cong M$ for all $M \in A\text{-mod}$. In particular, $(L^\lambda)^*$ is an irreducible module for all $\lambda \in \Lambda$. Furthermore, $(I^\lambda)^*$ and $(P^\lambda)^*$, $\lambda \in \Lambda$, are the projective cover and the injective envelope of $(L^\lambda)^*$, respectively.

2.4. Lemma. Assume that the K -algebra A equips an K -linear anti-involution $*$ satisfying $(L^\lambda)^* \cong L^\lambda$ for any $\lambda \in \Lambda$. Then the map $'$ defines an involution on the set Λ_0 . Moreover, for any $\lambda \in \Lambda_0$, we have

$$P^\lambda \cong I^{\lambda'} \cong (P^{\lambda'})^* \quad \text{and} \quad P^{\lambda'} \cong I^\lambda \cong (P^\lambda)^*.$$

Proof. Since $(L^\lambda)^* \cong L^\lambda$ for any $\lambda \in \Lambda$, we have $(I^\lambda)^* \cong P^\lambda$ and $(P^\lambda)^* \cong I^\lambda$. Therefore $P^{\lambda'} \cong (I^{\lambda'})^* \cong (P^\lambda)^*$ for all $\lambda \in \Lambda_0$ and hence $P^{\lambda'}$ is projective and injective for all $\lambda \in \Lambda_0$. It follows that $\lambda' \in \Lambda_0$ and $(\lambda')' = \lambda$ for any $\lambda \in \Lambda_0$ since $(M^*)^* \cong M$ for all $M \in A\text{-mod}$. Thus the map $'$ defines an involution on the set Λ_0 . This completes the proof. \square

From now on, we assume that the map $'$ defines an involution on Λ_0 throughout this section. Thus we have $\text{soc}(P^{\lambda'}) = \text{soc}(I^\lambda) \cong L^\lambda$ and hence $\text{Hom}_A(P^\lambda, \text{soc}(P^{\lambda'}))$ is one-dimensional by (2.1). Therefore for each $\lambda \in \Lambda_0$ the subspace of all the homomorphisms $f \in \text{Hom}_A(P^\lambda, P^{\lambda'})$ satisfying $\text{Im}(f) = \text{soc}(P^{\lambda'})$ is one-dimensional. For each $\lambda \in \Lambda_0$, there is a unique (up to a scalar) nonzero homomorphism

$$(2.5) \quad \theta_\lambda \in \text{Hom}_A(P^\lambda, P^{\lambda'}) \quad \text{satisfying} \quad \text{Im}(\theta_\lambda) = \text{soc}(P^{\lambda'}).$$

We will fix a θ_λ for each $\lambda \in \Lambda_0$.

2.6. Definition. Assume that the map $'$ defines an involution on Λ_0 . A K -basis Υ of B is said to be appropriate if

$$\bigcup_{\lambda, \mu \in \Lambda_0} \text{Hom}_A(P^\lambda, P^\mu) \supseteq \Upsilon \supseteq \{\theta_\lambda \mid \lambda \in \Lambda_0\}.$$

Given an appropriate K -basis (2.7) of B of the following form

$$(2.7) \quad \Upsilon := \bigcup_{\lambda, \mu \in \Lambda_0} \left\{ f_j^{\lambda, \mu} \mid f_j^{\lambda, \mu} \in \text{Hom}_A(P^\lambda, P^\mu), 1 \leq j \leq \dim \text{Hom}_A(P^\lambda, P^\mu) \right\},$$

we define a form tr on B determined by

$$(2.8) \quad \text{tr}(f_j^{\lambda, \mu}) := \begin{cases} 1, & \text{if } \mu = \lambda' \text{ and } f_j^{\lambda, \mu} = \theta_\lambda; \\ 0, & \text{if } \mu \neq \lambda', \text{ or } \mu = \lambda' \text{ and } f_j^{\lambda, \mu} \neq \theta_\lambda. \end{cases}$$

2.9. Definition. The form tr defined in (2.8) is called the canonical form attached to the appropriate basis Υ of B .

A K -linear map τ from a K -algebra R to K is called a form on R . The form τ induces an associative bilinear form $(-, -)_\tau$ on R defined by $(f, g)_\tau := \tau(fg)$ for all $f, g \in R$. Recall that a bilinear form $(-, -)$ on R is called associative if $(fh, g) = (f, hg)$ for all $f, g, h \in R$. The form τ is called symmetric (resp., non-degenerate) if the bilinear form $(-, -)_\tau$ is symmetric (resp. non-degenerate). The form τ is called a symmetrizing form on R if the bilinear form $(-, -)_\tau$ is a non-degenerate symmetric form. R is called a symmetric algebra if R equips with a symmetrizing form.

2.10. Lemma. Assume that the map $'$ defines an involution on Λ_0 . For $\lambda, \mu \in \Lambda_0$ and $0 \neq f \in \text{Hom}_A(P^\lambda, P^\mu)$, there exist $g \in \text{Hom}_A(P^\mu, P^{\lambda'})$ and $h \in \text{Hom}_A(P^{\mu'}, P^\lambda)$ such that $gf = \theta_\lambda$ and $fh = \theta_{\mu'}$.

Proof. Set $N := f^{-1}(\text{soc}(P^\mu))$. The restriction map $f|_N : N \rightarrow \text{soc}(P^\mu)$ of f on N is surjective because the image of the nonzero homomorphism f contains $\text{soc}(P^\mu) \cong L^{\mu'}$. Since $P^{\mu'}$ is the projective cover of $\text{soc}(P^\mu) \cong L^{\mu'}$ and $f|_N$ is an epimorphism, we can find a homomorphism $h_1 : P^{\mu'} \rightarrow N$ such that the image of $f|_N h_1$ is $\text{soc}(P^{\mu'})$. We may regard h_1 is a homomorphism from $P^{\mu'}$ to P^λ and hence $fh_1 = c\theta_{\mu'}$ for some nonzero $c \in K$. Therefore we have $fh = \theta_{\mu'}$ by choosing $h = c^{-1}h_1$.

Dually, we set $M := f(P^\lambda)$. We have a natural embedding $\iota : M \hookrightarrow P^\mu$. Since P^λ has a unique simple head L^λ , it follows that M has a unique simple head L^λ too. Let $\pi : M \rightarrow P^{\lambda'}$ be a homomorphism which sends M onto the unique simple socle L^λ of $P^{\lambda'}$. Since $P^{\lambda'}$ is injective, we can find a homomorphism $g \in \text{Hom}_A(P^\mu, P^{\lambda'})$ such that $g\iota = \pi$. Therefore $gf = \pi f = c\theta_\lambda$ for some nonzero $c \in K$. Now we take $g = c^{-1}g$, then $gf = \theta_\lambda$ as required. This completes the proof of the lemma. \square

2.11. Proposition. *If the map $'$ defines an involution on Λ_0 , then the bilinear form $(-, -)_{\text{tr}}$ induced by tr is non-degenerate. In other words, B is a Frobenius algebra over K . In particular, if the K -algebra A equips an K -linear anti-involution $*$ satisfying $(L^\lambda)^* \cong L^\lambda$ for any $\lambda \in \Lambda$, then B is a Frobenius algebra over K .*

Proof. Let $f = \sum_{\lambda, \mu \in \Lambda_0} f_{\lambda, \mu}$ be a nonzero element in B where $f_{\lambda, \mu} \in \text{Hom}_A(P^\lambda, P^\mu)$ for all $\lambda, \mu \in \Lambda_0$. Let $g := f_{\lambda, \mu}$ be a nonzero component of f . By Lemma 2.10, we can find a homomorphism $h : P^{\mu'} \rightarrow P^\lambda$ such that $gh = \theta_{\mu'}$. Therefore we have

$$(f, h)_{\text{tr}} = \text{tr}\left(\left(\sum_{\gamma, \beta \in \Lambda_0} f_{\gamma, \beta}\right)h\right) = \text{tr}\left(\left(\sum_{\beta \in \Lambda_0} f_{\lambda, \beta}\right)h\right) = \text{tr}(f_{\lambda, \mu}h) = \text{tr}(\theta_{\mu'}) = 1 \neq 0.$$

The third equality follows from the definition (2.8) of the form tr . The second follows from Lemma 2.4. \square

In general, we do not know whether the bilinear form $(-, -)_{\text{tr}}$ on B is symmetric or not. The following lemma gives necessary conditions for a form to be a symmetrizing form on B .

2.12. Lemma. *Assume the map $'$ defines an involution on Λ_0 . If there exists a symmetrizing form τ on B , then we have*

- (i) $\lambda' = \lambda$ and $\tau(\theta_\lambda) \neq 0$ for all $\lambda \in \Lambda_0$;
- (ii) $\tau(f) = 0$ for all $f \in \text{Hom}_A(P^\lambda, P^\mu)$ with $\lambda \neq \mu \in \Lambda_0$.

Proof. Since τ is non-degenerate, for each $\gamma \in \Lambda_0$ we can find a nonzero element $f = \sum_{\lambda, \mu \in \Lambda_0} f_{\lambda, \mu}$ in B such that $(\theta_\gamma, f)_\tau \neq 0$, where $f_{\lambda, \mu} \in \text{Hom}_A(P^\lambda, P^\mu)$ for all $\lambda, \mu \in \Lambda_0$. Since $\tau(\theta_\gamma f) \neq 0$, we have

$$0 \neq \theta_\gamma f = \theta_\gamma \left(\sum_{\lambda, \mu \in \Lambda_0} f_{\lambda, \mu} \right) = \theta_\gamma \left(\sum_{\lambda \in \Lambda_0} f_{\lambda, \gamma} \right) = \theta_\gamma f_{\gamma, \gamma}.$$

The last equality follows from the fact that there is no nonzero homomorphism from P^λ to $\theta_\gamma(P^\gamma) \cong L^\gamma$ for all $\lambda \neq \gamma$. Since $\theta_\gamma f_{\gamma, \gamma}$ is a nonzero homomorphism with image contained in $\text{soc}(P^{\gamma'}) \cong L^\gamma$, we have $\theta_\gamma f_{\gamma, \gamma} = c\theta_\gamma$ for some nonzero $c \in K$. Therefore

$$\tau(\theta_\gamma) = c^{-1}\tau(\theta_\gamma f_{\gamma, \gamma}) = c^{-1}\tau(\theta_\gamma f) = c^{-1}(\theta_\gamma, f)_\tau \neq 0.$$

Since τ is symmetric, we have

$$0 \neq \tau(\theta_\gamma) = \tau(\text{id}_{P^{\gamma'}} \theta_\gamma \text{id}_{P^\gamma}) = \tau(\text{id}_{P^\gamma} \text{id}_{P^{\gamma'}} \theta_\gamma).$$

Therefore we have $\gamma' = \gamma$ for all $\gamma \in \Lambda_0$.

Finally, for any $\lambda, \mu \in \Lambda_0$ with $\lambda \neq \mu$ and $f \in \text{Hom}_A(P^\lambda, P^\mu)$, we have

$$\tau(f) = \tau(\text{id}_{P^\mu} f \text{id}_{P^\lambda}) = \tau(\text{id}_{P^\lambda} \text{id}_{P^\mu} f) = \tau(0) = 0$$

since τ is symmetric. This completes the proof of the lemma. \square

For each A -module M , we define $\text{rad}^0(M) = M$ and define the radical filtration on M inductively by $\text{rad}^i(M) = \text{rad}(\text{rad}^{i-1}(M))$. Note that $\text{rad}^i(M) = (\text{rad } A)^i M$, where $\text{rad}(A)$ is the Jacobson radical of A .

Note that in Lemma 2.10 we do not know whether we can take g to be h or not in there such that $gf = \theta_\lambda$ and $fh = \theta_\mu$. The following lemma ensures that the expectation holds for the existence of a symmetrizing form τ on B satisfying $\tau(\theta_\lambda) = 1$ for all $\lambda \in \Lambda_0$.

2.13. Proposition. *Assume the map $'$ defines an involution on Λ_0 and there exists a symmetrizing form τ on B such that $\tau(\theta_\lambda) = 1$ for all $\lambda \in \Lambda_0$. For any nonzero $f \in \text{Hom}_A(P^\lambda, P^\mu)$ with $\lambda, \mu \in \Lambda_0$, there exists a homomorphism $g \in \text{Hom}_A(P^\mu, P^\lambda)$ such that $fg = \theta_\mu$ and $gf = \theta_\lambda$.*

Proof. By Lemma 2.12, we have $\lambda' = \lambda$ for all $\lambda \in \Lambda_0$. By Lemma 2.10, there is a homomorphism $h \in \text{Hom}_A(P^\mu, P^\lambda)$ satisfying $fh \neq 0$. Let s be the maximal integer such that there is a homomorphism $g \in \text{Hom}_A(P^\mu, \text{rad}^s(P^\lambda))$ satisfying $fg \neq 0$. We claim that $fg \in K^\times \theta_\mu$.

Suppose that $g \in \text{Hom}_A(P^\mu, \text{rad}^s(P^\lambda))$ and $fg \notin K^\times \theta_\mu$. Then $L^\mu \cong \text{soc}(P^\mu) \subsetneq \text{Im}(fg)$. Applying Lemma 2.10, we can find $h \in \text{Hom}_A(P^\mu, P^\mu)$ such that $fgh = \theta_\mu$. Since $\text{soc}(P^\mu) \subsetneq \text{Im}(fg)$, we can deduce that h is not an isomorphism and hence h is not injective (because every injective endomorphism of P^μ is automatically an automorphism). It follows that $\text{Im}(h) \subseteq \text{rad}(P^\mu)$. By assumption that $g \in \text{Hom}_A(P^\mu, \text{rad}^s(P^\lambda))$, we have $\text{Im}(gh) \in \text{rad}^{s+1}(P^\lambda)$ and $f(gh) \neq 0$, we get a contradiction to the maximality of s . This proves our claim.

Now we may assume that $fg = \theta_\mu$. Note that $\tau(gf) = \tau(fg) = \tau(\theta_\mu) \neq 0$ by Lemma 2.12. It follows that $gf \neq 0$. By Lemma 2.10 again we can find a homomorphism $h \in \text{Hom}_A(P^\lambda, P^\lambda)$ such that $gh = \theta_\lambda$. It follows that $\tau(fhg) = \tau(gfh) = \tau(\theta_\lambda) \neq 0$ by Lemma 2.12. In particular, $fgh \neq 0$. We claim that h is an isomorphism. Otherwise, h is not injective (because every injective endomorphism of P^μ is automatically an automorphism) and hence $\text{Im}(h) \subseteq \text{rad}(P^\lambda)$. It follows that for all $i \geq 0$,

$$h(\text{rad}^i(P^\lambda)) = h(\text{rad}^i(A)P^\lambda) = \text{rad}(A)^i h(P^\lambda) \subseteq \text{rad}(A)^i \text{rad}(P^\lambda) = \text{rad}^{i+1}(P^\lambda).$$

Then $hg(P^\lambda) \subseteq h(\text{rad}^s(P^\lambda)) \subseteq \text{rad}^{s+1}(P^\lambda)$. We get a contradiction to our assumption because $hg(P^\lambda) \subseteq \text{rad}^{s+1}(P^\lambda)$ and $f(hg) \neq 0$. This proves our claim. Therefore h is an isomorphism and $gf \in K^\times \theta_\lambda$. Since $\tau(gf) = \tau(fg)$ and $\tau(\theta_\lambda) = 1$ for any $\lambda \in \Lambda_0$, it follows that $gf = \theta_\lambda$. \square

3. GRADED ALGEBRAS

We are interested in searching for conditions to ensure the canonical form tr attached to a given appropriate basis of the endomorphism algebra of any projective-injective module over any finite-dimensional algebra A is a symmetrizing form. In this section, we consider the \mathbb{Z} -graded finite-dimensional algebras and give a sufficient condition, called an admissible condition (see Definition 3.7 below), for the appropriate basis of the endomorphism algebra $B = \text{End}_A\left(\bigoplus_{\lambda \in \Lambda_0} P^\lambda\right)$ (see (2.3)) of the basic projective-injective module such that the canonical form tr attached to it is symmetric. For a \mathbb{Z} -graded finite-dimensional algebra A , B is a symmetric algebra if and only if there exists an admissible K -basis of B , and hence the canonical form tr attached to the admissible basis is a symmetrizing form, see Corollary 3.12 and Proposition 3.9 below.

For a field K , a graded K -vector space M means a \mathbb{Z} -graded K -vector space $M = \bigoplus_{k \in \mathbb{Z}} M_k$ such that M_d are finite-dimensional K -subspaces of M for all $d \in \mathbb{Z}$. For $d \in \mathbb{Z}$ and $v \in M_d$, v is called a homogeneous element of degree d and we write $\deg v = d$. For $d \in \mathbb{Z}$ and graded K -vector spaces $M = \bigoplus_{k \in \mathbb{Z}} M_k$ and $N = \bigoplus_{k \in \mathbb{Z}} N_k$, $\text{Hom}_K(M, N) = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_K(M, N)_d$ forms a graded K -vector space, where $\text{Hom}_K(M, N)_d := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_K(M_k, N_{k+d})$ denotes the subspace spanned by homogeneous linear maps of degree d in $\text{Hom}_K(M, N)$ for all $d \in \mathbb{Z}$.

A graded K -algebra means a finite-dimensional unital associative K -algebra R such that $R = \bigoplus_{d \in \mathbb{Z}} R_d$ is a graded K -vector space satisfying $R_d R_k \subset R_{d+k}$, for all $d, k \in \mathbb{Z}$. It follows that $1 \in R_0$. A graded K -algebra R is called positively graded if $R = \bigoplus_{d \in \mathbb{Z}_+} R_d$. A graded (left) R -module is a graded finite-dimensional K -vector space $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that M is an R -module and $R_k M_d \subset M_{d+k}$, for all $d, k \in \mathbb{Z}$. For a graded K -module M and $k \in \mathbb{Z}$, let $M\langle k \rangle$ be the graded K -module obtained by shifting the grading on M up by k . That is, $M\langle k \rangle_d := M_{d-k}$, for $d \in \mathbb{Z}$. For graded R -modules M and N and $d \in \mathbb{Z}$, $f \in \text{Hom}_R(M, N)$ is called a homogeneous homomorphism of degree d if $f(M_k) \subset N_{d+k}$ for all $k \in \mathbb{Z}$. Let $\text{Hom}_R(M, N)_d$ denote the subspace of $\text{Hom}_R(M, N)$ consisting of homogeneous homomorphisms of degree d . Then $\text{Hom}_R(M, N)$ forms a graded K -vector space and $\text{Hom}_R(M, N) = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_R(M, N)_d$. For graded R -modules M and N , let

$$\text{hom}_R(M, N) := \text{Hom}_R(M, N)_0.$$

and let $M \simeq N$ denote that there is a homogeneous isomorphism of degree 0 between M and N . The graded length of $M \in R\text{-gmod}$ is defined to be

$$(3.1) \quad b - a, \quad \text{for } M = \bigoplus_{i=a}^b M_i \text{ with } M_a \neq 0 \neq M_b.$$

In particular, the graded length of M is b if $M = \bigoplus_{i=0}^b M_i$ with $M_0 \neq 0 \neq M_b$.

For a graded K -algebra R , let $R\text{-gmod}$ denote the category of graded finite-dimensional R -modules with homomorphisms of degree 0. The forgetful functor $R\text{-gmod} \rightarrow R\text{-mod}$ is denoted by $\underline{\text{For}}$. An R -module M is called gradable if $M \cong \underline{\text{For}}(N)$ for some $N \in R\text{-gmod}$. In this case, M is also said to have a graded lift. For an indecomposable module $M \in R\text{-gmod}$, a module $N \in R\text{-gmod}$ and $j, k \in \mathbb{Z}$, let $\sigma_{j,k}$ denote isomorphism of K -vector spaces

$$(3.2) \quad \begin{aligned} \sigma_{j,k} : \text{Hom}_R(M, N) &\longrightarrow \text{Hom}_R(M\langle j \rangle, N\langle k \rangle) \\ f &\mapsto f[j, k] := \sigma_{j,k}(f) \end{aligned}$$

such that $\underline{\text{For}}(f[j, k]) = \underline{\text{For}}(f)$ for all $f \in \text{Hom}_R(M, N)$. We also let

$$(3.3) \quad f[j] := f[j, j], \quad \text{for } f \in \text{Hom}_R(M, N) \text{ and } j \in \mathbb{Z}.$$

Note that

$$\sigma_{j,k}(\text{Hom}_R(M, N)_d) = \text{Hom}_R(M\langle j \rangle, N\langle k \rangle)_{d-j+k} \quad \text{for all } d \in \mathbb{Z}.$$

We will adapt the notations and assumptions defined in Section 2. Recall that A is a finite-dimensional algebra over the algebraically closed field K . In the section, we will further assume that A is a positively graded K -algebra. Since A is positively graded, every simple module in $A\text{-gmod}$ concentrates in a fixed degree. Therefore every simple A -module is gradable. For each $\lambda \in \Lambda$, we fix a \mathbb{Z} -grading on L^λ , is also denoted by L^λ , by defining L^λ concentrated in degree 0. Recall that $\{L^\lambda | \lambda \in \Lambda\}$ denote a complete set of pairwise non-isomorphic simple A -modules. Then $\{L^\lambda\langle k \rangle | \lambda \in \Lambda, k \in \mathbb{Z}\}$ forms a complete set of pairwise non-isomorphic graded simple A -modules.

For each $\lambda \in \Lambda$, the projective cover P^λ of L^λ is gradable (see, for example, [6, Corollary 3.4]). Note that the natural homomorphism from P^λ to L^λ is of degree 0. Therefore we can write P^λ in the following form

$$(3.4) \quad P^\lambda = \bigoplus_{i=0}^{d_\lambda} P_i^\lambda \quad \text{such that } P_{d_\lambda}^\lambda \neq 0 \text{ for all } \lambda \in \Lambda.$$

As a consequence, $B = \text{End}_A\left(\bigoplus_{\lambda \in \Lambda_0} P^\lambda\right)$ is a positively graded algebra.

3.5. Lemma. *Assume the map $'$ defines an involution on Λ_0 . For $\lambda, \mu \in \Lambda_0$ and a nonzero $f \in \text{Hom}_A(P^\lambda, P^\mu)_j$, there exist $g \in \text{Hom}_A(P^\mu, P^{\lambda'})_{d_{\lambda'}-j}$ and $h \in \text{Hom}_A(P^{\mu'}, P^\lambda)_{d_\mu-j}$ such that $gf = \theta_\lambda$ and $fh = \theta_{\mu'}$.*

Proof. By Lemma 2.10, there exists $g \in \text{Hom}_A(P^\mu, P^{\lambda'})$ such that $gf = \theta_\lambda$. Write $g = \sum_i g_i$, where $g_i \in \text{Hom}_A(P^\mu, P^{\lambda'})_i$ for all i . Then we have $g_{d_{\lambda'}-j}f = \theta_\lambda$. A similar proof shows the existence of h . \square

For each $\lambda \in \Lambda_0$, the homomorphism θ_λ defined in (2.5) is clearly a homogeneous element in B since $\text{soc}(P^{\lambda'})$ is a graded submodule of $P^{\lambda'}$ [6, Theorem 3.5]. By (3.4), the degree of θ_λ in B is

$$(3.6) \quad \deg \theta_\lambda = d_{\lambda'} \quad \text{for every } \lambda \in \Lambda_0.$$

3.7. Definition. Let A be a positively graded K -algebra such that $\text{soc}(P^\lambda) \simeq L^\lambda\langle d_\lambda \rangle$ for each $\lambda \in \Lambda_0$ (i.e. $\lambda' = \lambda$ for each $\lambda \in \Lambda_0$). An appropriate basis Υ of $B = \text{End}_A\left(\bigoplus_{\lambda \in \Lambda_0} P^\lambda\right)$ consisting of homogeneous elements in B is said to satisfy the admissible conditions if for each $\lambda, \mu \in \Lambda_0$ and $j \in \mathbb{Z}$ with $\Upsilon \cap \text{Hom}_A(P^\lambda, P^\mu)_j \neq 0$, then the following conditions hold:

- (i) $d_\mu = d_\lambda$, where d_γ is defined in (3.4);
- (ii) $|\Upsilon \cap \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-j}| = |\Upsilon \cap \text{Hom}_A(P^\lambda, P^\mu)_j|$;
- (iii) for any $f \in \Upsilon \cap \text{Hom}_A(P^\lambda, P^\mu)_j$, there is exactly one element $g \in \Upsilon \cap \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-j}$ such that $gf = c_{gf}\theta_\lambda$, $fg = c_{fg}\theta_\mu$ for some $c_{fg} = c_{gf} \in K^\times$, and $hf = fh = 0$ for all $h \in \Upsilon \cap \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-j}$ with $h \neq g$.

A basis of B is called admissible if it is an appropriate basis satisfying the admissible conditions.

3.8. Remark. The condition (i) means that every P^λ with $\lambda \in \Lambda_0$ in the same block has the same graded length (see the discussion above Proposition 3.22). The condition (ii) in the definition follows from the condition (iii). The condition (iii) implies that for all $f \in \text{Hom}_A(P^\lambda, P^\mu)_j$, $g \in \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-j}$ with $\lambda, \mu \in \Lambda_0$ and for all $j \in \mathbb{Z}$, we have

$$fg = c\theta_\mu \quad \text{for some } c \in K^\times \quad \Leftrightarrow \quad gf = c\theta_\lambda.$$

The following proposition follows easily from Remark 3.8 and the definition of the canonical form tr attached to an admissible basis defined in (2.8).

3.9. Proposition. *Assume that A be a positively graded algebra with an admissible basis Υ . If $\text{soc}(P^\mu) \simeq L^\mu \langle d_\mu \rangle$ for each $\mu \in \Lambda_0$, then the canonical form tr attached to the admissible basis Υ is a symmetrizing form on B . In particular, B is a symmetric algebra over K .*

3.10. Proposition. *Assume A is a positively graded algebra and $P_{d_\lambda}^\lambda = \text{soc}(P^\lambda)$ for all $\lambda \in \Lambda_0$. If there exists a symmetrizing form τ on B , then there exists an admissible basis of B .*

Proof. By Lemma 2.12, we have $\lambda' = \lambda$ and $\tau(\theta_\lambda) \neq 0$ for all $\lambda \in \Lambda_0$. We may choose θ_λ s such that $\tau(\theta_\lambda) = 1$ for all $\lambda \in \Lambda_0$. Using Lemma 3.17 and the assumption, we have $P^\lambda = \bigoplus_{i=0}^{d_\lambda} P_i^\lambda$ such that $P_{d_\lambda}^\lambda = \text{soc } P^\lambda = L^\lambda \langle d_\lambda \rangle$ for all $\lambda \in \Lambda_0$.

Fix $\lambda, \mu \in \Lambda_0$ and $k \in \mathbb{Z}$ such that $\text{Hom}_A(P^\lambda, P^\mu)_k \neq 0$. We claim that $d_\lambda = d_\mu$. Let $f \in \text{Hom}_A(P^\lambda, P^\mu)_k$ be a nonzero homomorphism. By Lemma 3.5, there exist $g \in \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k}$ and $h \in \text{Hom}_A(P^\mu, P^\lambda)_{d_\mu-k}$ such that $gf = \theta_\lambda$ and $fh = \theta_\mu$. Therefore $\tau(fg) = \tau(gf) = \tau(\theta_\lambda) \neq 0$ and $\tau(hf) = \tau(fh) = \tau(\theta_\mu) \neq 0$ by Lemma 2.12. Hence $fg \neq 0 \neq hf$, and

$$d_\lambda = \deg(gf) = \deg(fg) \leq d_\mu, \quad \text{and} \quad d_\mu = \deg(fh) = \deg(hf) \leq d_\lambda.$$

This implies $d_\lambda = d_\mu$ and $\text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k} \neq 0$.

Now we will show that there are bases of $\text{Hom}_A(P^\lambda, P^\mu)_k$ and $\text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k}$ satisfying the conditions (ii) and (iii) of Definition 3.7 for the case $\text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k} \neq \text{Hom}_A(P^\lambda, P^\mu)_k \neq 0$. By Lemma 3.5, there is a non-degenerate pairing from $\text{Hom}_A(P^\lambda, P^\mu)_k \times \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k}$ to $K\theta_\mu$ defined by (f, g) sending to fg . Then we have $\dim \text{Hom}_A(P^\lambda, P^\mu)_k = \dim \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k}$, and a basis $\{f_1, \dots, f_m\}$ of $\text{Hom}_A(P^\lambda, P^\mu)_k$ and a basis $\{g_1, \dots, g_m\}$ of $\text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k}$ such that $f_i g_j = \delta_{ij} \theta_\mu$ for all i, j . Since $P_{d_\lambda}^\lambda = \text{soc}(P^\lambda)$ and the degree of $g_j f_i$ equals $d_\mu = d_\lambda$ for all i, j , we have $g_j f_i \in K\theta_\lambda$ for all i, j . Let $g_j f_i = c_{ji} \theta_\lambda$. Then

$$c_{ji} = \tau(c_{ji} \theta_\lambda) = \tau(g_j f_i) = \tau(f_i g_j) = \tau(\delta_{ij} \theta_\mu) = \delta_{ij}.$$

Therefore $f_i g_j = \delta_{ij} \theta_\mu$ and $g_j f_i = \delta_{ij} \theta_\lambda$ for all i, j .

Now we consider the case $\text{Hom}_A(P^\lambda, P^\mu)_k = \text{Hom}_A(P^\mu, P^\lambda)_{d_\lambda-k}$ with $\text{Hom}_A(P^\lambda, P^\mu)_k \neq 0$. Then we have $\lambda = \mu$, $k = d_\lambda - k$ and a non-degenerate pairing from $\text{Hom}_A(P^\lambda, P^\lambda)_k \times \text{Hom}_A(P^\lambda, P^\lambda)_k$ to $K\theta_\lambda$ defined by (f, g) sending to fg . By some standard arguments, there is a basis $\{f_1, \dots, f_m\}$ of $\text{Hom}_A(P^\lambda, P^\lambda)_j$ such that for each j there is unique j' satisfying $f_j f_{j'} = \theta_\lambda$ and $f_j f_i = 0$ for all $i \neq j'$. Using the same argument as above, we have $f_{j'} f_j = f_j f_{j'} = \theta_\lambda$ and $f_i f_j = f_j f_i = 0$ for $i \neq j'$.

Taking the union of all bases obtained from above, we get an admissible basis of B . \square

3.11. Remark. If A is a positively graded algebra such that A_0 is a semisimple K -algebra, then the assumption that $P_{d_\lambda}^\lambda = \text{soc}(P^\lambda)$ for all $\lambda \in \Lambda_0$ automatically holds.

The following corollary is a consequence of Proposition 3.9 and Proposition 3.10.

3.12. Corollary. *Assume A is a positively graded algebra such that $P_{d_\lambda}^\lambda = \text{soc}(P^\lambda) \simeq L^\lambda \langle d_\lambda \rangle$ for each $\lambda \in \Lambda_0$. Then B is a symmetric algebra if and only if there exists an admissible K -basis of B .*

3.13. Definition. Let R be a graded K -algebra. A form $\tau : R \rightarrow K$ is called homogeneous if τ is a homogeneous map, where K is regarded as a graded vector space concentrated in degree 0. If $\tau : M \rightarrow K$ is a homogeneous form, then the associated bilinear form $(-, -)_\tau$ is also called a homogeneous bilinear form on R .

For a projective-injective A -module Q , Q is gradable and $Q \simeq \bigoplus_{\mu \in \Lambda_0} (P^\mu)^{\oplus k_\mu}$ for some $k_\mu \in \mathbb{Z}_+$ such that not all k_μ s are zero.

3.14. Corollary. *Assume that A is a positively graded algebra and $P_{d_\lambda}^\lambda = \text{soc}(P^\lambda) \simeq L^\lambda \langle d_\lambda \rangle$ for each $\lambda \in \Lambda_0$. Let Q be any projective-injective A -module. If B is a symmetric algebra over K , then $\text{End}_A(Q)$ is a symmetric algebra over K . Moreover, if the projective-injective A -module $Q = \bigoplus_{\mu \in \Lambda_0} (P^\mu)^{\oplus k_\mu}$ such that $d_\mu = d$ for all $k_\mu \neq 0$, then there is a homogeneous symmetrizing form on $\text{End}_A(Q)$ of degree $-d$.*

Proof. By Lemma 2.12, we have $\lambda' = \lambda$ and $\tau(\theta_\lambda) \neq 0$ for all $\lambda \in \Lambda_0$. By Proposition 3.10 and Proposition 3.9, there exists an admissible basis of B and tr is a symmetrizing form on B .

Let $Q = \bigoplus_{\mu \in \Lambda_0} (P^\mu)^{\oplus k_\mu}$ for some $k_\mu \in \mathbb{Z}_+$ and let $\widehat{B} := \text{End}_A(Q)$. Since P^μ are graded A -module for all $\mu \in \Lambda_0$, \widehat{B} is a positively graded K -algebra. For $\mu \in \Lambda_0$ and $s \in \mathbb{Z}_{>0}$ such that $1 \leq s \leq k_\mu$,

let $(P^\mu)^{(s)}$ denote the s -th component of $(P^\mu)^{\oplus k_\mu}$. We regard that $(P^\mu)^{(s)} \subseteq (P^\mu)^{\oplus k_\mu} \subseteq Q$ and $\text{Hom}_A((P^\mu)^{(s)}, (P^\lambda)^{(t)}) \subseteq \widehat{B}$ in the natural way for $\mu, \lambda \in \Lambda_0$, $1 \leq s \leq k_\mu$ and $1 \leq t \leq k_\lambda$. For $\mu, \lambda \in \Lambda_0$, $1 \leq s \leq k_\mu$ and $1 \leq t \leq k_\lambda$, there is an isomorphism of K -vector space from $\text{Hom}_A(P^\mu, P^\lambda)$ to $\text{Hom}_A((P^\mu)^{(s)}, (P^\lambda)^{(t)})$ by sending f to $f^{(s,t)}$ defined in the obvious way. Define a form on \widehat{B} determined by

$$\widehat{\text{tr}}(f) := \begin{cases} \text{tr}(g), & \text{if } f \in \text{Hom}_A((P^\mu)^{(s)}, (P^\lambda)^{(t)}) \text{ such that } f = g^{(s,t)} \text{ with } s = t; \\ 0, & \text{if } f \in \text{Hom}_A((P^\mu)^{(s)}, (P^\lambda)^{(t)}) \text{ with } s \neq t. \end{cases}$$

It is clear that $\widehat{\text{tr}}$ is a symmetrizing form on \widehat{B} by Proposition 3.10, Proposition 3.9 and Remark 3.8. Finally, it is obvious that $\widehat{\text{tr}}$ is a homogeneous linear map on $\text{End}_A(Q)$ of degree d if $d_\mu = d$ for all $k_\mu \neq 0$. \square

For the rest of this section, we will further assume that A is a positively graded K -algebra equipped with a homogeneous anti-involution \star of degree 0 satisfying $(L^\lambda)^\circ \simeq L^\lambda$ (defined below) for each $\lambda \in \Lambda$. Let $*$ denote the anti-involution \star of A without considering the gradation. Then the dual of the graded A -module M is the graded A -module

$$M^\circ = \bigoplus_{j \in \mathbb{Z}} M_j^\circ \quad \text{where } M_j^\circ := \text{Hom}_K(M_{-j}, K)$$

and the action of A on M° is given by $(af)(m) = f(a^*m)$ for all $f \in M^\circ$, $a \in A$ and $m \in M$. It is clear that

$$(3.15) \quad (M^\circ)^\circ \simeq M \quad \text{and} \quad (M\langle k \rangle)^\circ \simeq M^\circ\langle -k \rangle \quad \forall M \in A\text{-gmod}, k \in \mathbb{Z}.$$

It is clear that \circ gives an equivalence of categories. Recall the simple modules L^λ lie in $A\text{-gmod}$ concentrating in the degree zero. Therefore $(L^\lambda)^\circ$ are simple modules concentrating in the degree zero for all $\lambda \in \Lambda$. The assumptions say

$$(3.16) \quad (L^\lambda)^\circ \simeq L^\lambda \quad \text{for each } \lambda \in \Lambda.$$

and hence $L^\lambda\langle k \rangle^\circ \simeq L^\lambda\langle -k \rangle$ for each $\lambda \in \Lambda$ and $k \in \mathbb{Z}$.

3.17. Lemma. *Assume that A is a positively graded algebra. For each $\lambda \in \Lambda_0$, we have $\text{soc}(P^\lambda) \subseteq P_{d_\lambda}^\lambda$. Assume further that A equips with a homogeneous anti-involution \star of degree 0 satisfying $(L^\lambda)^\circ \simeq L^\lambda$ for each $\lambda \in \Lambda$ such that $\text{soc}(P^\lambda) \simeq L^\lambda\langle d_\lambda \rangle$ for each $\lambda \in \Lambda_0$. Then we have*

$$(3.18) \quad (P^\lambda)^\circ \simeq P^\lambda\langle -d_\lambda \rangle \quad \text{for each } \lambda \in \Lambda_0.$$

Proof. $P_{d_\lambda}^\lambda$ is a graded A -submodule of P^λ since A is positively graded. Therefore $P_{d_\lambda}^\lambda$ contains the simple socle $\text{soc } P^\lambda$ of P^λ . To show (3.18), there is an injective map $g \in \text{hom}_A(L^\lambda\langle d_\lambda \rangle, P^\lambda)$ since $\text{soc}(P^\lambda) \subseteq P_{d_\lambda}^\lambda$. Therefore there is a surjective map $h \in \text{hom}_A((P^\lambda)^\circ, L^\lambda\langle -d_\lambda \rangle)$ because $(L^\lambda\langle d_\lambda \rangle)^\circ \simeq L^\lambda\langle -d_\lambda \rangle$. Now $(P^\lambda)^\circ \simeq P^\lambda\langle -d_\lambda \rangle$ follows from the fact that $\underline{\text{For}}((P^\lambda)^\circ)$ is a projective cover of L^λ in $A\text{-mod}$. \square

3.19. Definition. ([5, 14]) Let A be a finite-dimensional positively graded algebra over an algebraically closed field K . Let $\mathcal{C} := A\text{-gmod}$ be the category of graded finite-dimensional A -modules. Let \star be a homogenous anti-involution of degree zero of A which induces a graded duality functor \circ on \mathcal{C} . Let (Λ, \leq) be a finite poset and let $\Delta := \{\Delta^\lambda | \lambda \in \Lambda\}$ be a family of objects in \mathcal{C} . The pair (\mathcal{C}, Δ) is called a \mathbb{Z} -graded highest weight category with a duality functor \circ if

- (i) for each $\lambda \in \Lambda$, Δ^λ has a unique simple head L^λ satisfying $(L^\lambda)^\circ \cong L^\lambda$, and $\{L^\lambda\langle k \rangle | \lambda \in \Lambda, k \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic simple modules in \mathcal{C} ;
- (ii) $\text{Hom}_{\mathcal{C}}(\Delta^\lambda, M) = 0$ for each $\lambda \in \Lambda$ implies that $M = 0$;
- (iii) for each $\lambda \in \Lambda$, there is an indecomposable graded projective module P^λ in \mathcal{C} and a degree 0 surjection $f : P^\lambda \twoheadrightarrow \Delta^\lambda$ such that $\text{Ker}(f)$ has a finite filtration whose successive quotient are objects of the form $\Delta^\mu\langle k \rangle$ with $\mu > \lambda$ and $k \in \mathbb{Z}$;
- (iv) for each $\lambda \in \Lambda$, we have $\text{End}_{\mathcal{C}}(\Delta^\lambda) \cong K$;
- (v) for each $\lambda, \mu \in \Lambda$ such that $\text{Hom}_{\mathcal{C}}(\Delta^\lambda, \Delta^\mu) \neq 0$, we have $\lambda \leq \mu$.

Let v be an indeterminate over \mathbb{Z} . For $M \in A\text{-gmod}$ and a graded simple A -module L , the graded dimension of M is the Laurent polynomial

$$(3.20) \quad \dim_v M := \sum_{d \in \mathbb{Z}} (\dim_K M_d) v^d$$

and the graded multiplicity of L in M is the Laurent polynomial

$$(3.21) \quad [M : L]_v := \sum_{k \in \mathbb{Z}} [M : L\langle k \rangle] v^k.$$

For any $\lambda, \mu \in \Lambda$, the graded Cartan matrix $c_{\lambda, \mu}(v)$ is defined by

$$c_{\lambda, \mu}(v) := \dim_v \operatorname{Hom}_A(P^\mu, P^\lambda) = \sum_{k \in \mathbb{Z}} [P^\lambda : L^\mu\langle k \rangle] v^k.$$

For a finite-dimensional K -algebra R , two indecomposable R -modules M and N are said to be linked if either $\operatorname{Hom}_R(M, N) \neq 0$ or $\operatorname{Hom}_R(N, M) \neq 0$. R has a unique decomposition into a direct sum of indecomposable blocks. A block of R means an indecomposable two-sided ideal of R . It is well known that the equivalence relation on the simple R -modules induced from the block decomposition of R , where two simples are equivalent if they belong to the same block, coincides with the linkage classes of simple R -modules, where the equivalence relation is generated by $L^\lambda \sim L^\mu$ if $\operatorname{Ext}^1(L^\lambda, L^\mu) \neq 0$ or $\operatorname{Ext}^1(L^\mu, L^\lambda) \neq 0$, where L^λ and L^μ are simple R -modules. Also the linkage classes of simple R -modules coincides with the equivalence relation generated by $L^\lambda \sim L^\mu$ if $\operatorname{Hom}_R(P^\lambda, P^\mu) \neq 0$ or $\operatorname{Hom}_R(P^\mu, P^\lambda) \neq 0$, where P^λ and P^μ are the projective covers of the simple R -modules L^λ and L^μ , respectively. We also let $\operatorname{Hom}_{-R}(U, V)$ denote the K -vector space of R -homomorphism from the right R -module U to the right R -module V .

3.22. Proposition. *Let A be a positively graded algebra equipped with a homogeneous anti-involution \star of degree 0 such that $\operatorname{soc}(P^\lambda) \simeq L^\lambda\langle d_\lambda \rangle$ for all $\lambda \in \Lambda_0$. Assume that either (A, \star) is a \mathbb{Z} -graded cellular algebra in the sense of [7] or $(A\text{-gmod}, \otimes)$ is a \mathbb{Z} -graded highest weight category with a duality functor. If there is a homogenous idempotent element $e \in A$ of degree zero satisfying $Ae \simeq \bigoplus_{\lambda \in \Lambda_0} P^\lambda$ and $e^\star = e$, and there is a double centralizer property between A and $B^\circ := \left(\operatorname{End}_A(Ae)\right)^{\operatorname{op}}$ on Ae (i.e., the canonical map $A \rightarrow \operatorname{End}_{-B^\circ}(Ae)$ is an isomorphism), then $d_\lambda = d_\mu$ for any $\lambda, \mu \in \Lambda_0$ such that L^λ and L^μ belong in the same block.*

Proof. We will follow the definitions from [8, 14] closely in the proof of the proposition. Note that $B^\circ \simeq eAe$ since e is a homogenous idempotent element of degree zero. Let $B' = eAe$ and let $A = \bigoplus_{i=1}^m A_i$ and $B' = \bigoplus_{j=1}^n B'_j$ be the decompositions of A and B' into indecomposable blocks, respectively. Let e_1, \dots, e_m be the block idempotents of A . Let $F : A\text{-mod} \rightarrow B^\circ\text{-mod}$ be the Schur functor defined by: $F(M) := eM$, $F(f) : ev \mapsto ef(v)$, for all $M, N \in A\text{-mod}$, $f \in \operatorname{Hom}_A(M, N)$, $v \in M$. We first show that F induces a bijection between the indecomposable blocks of A and B' . The proof is the same as [8, Corollary 2.18]. For completeness, we include it here.

First we will show $m = n$. Let $Q := Ae$ and $Q_j := e_j Q$, for $1 \leq j \leq m$. Since $A \simeq \operatorname{End}_{B'}(Ae)$, Ae is a faithful left A -module. Then $Q_r \neq 0$ for $1 \leq r \leq m$, so $Q = \bigoplus_{i=1}^m Q_i$ is the decomposition of Q into its block components. In particular, $\operatorname{Hom}_A(Q_r, Q_s) = 0$ if $r \neq s$. Therefore, $B^\circ = \operatorname{End}_A(Q)^{\operatorname{op}} \simeq \bigoplus_{i=1}^m \operatorname{End}_A(Q_i)^{\operatorname{op}}$ is a decomposition of B° into (not necessarily indecomposable) blocks. In particular, $n \geq m$.

On the other hand, the assumptions that $e^\star = e$ and the isomorphism $A \simeq \operatorname{End}_{B'}(Ae)$ imply that the canonical map

$$(3.23) \quad A^{\operatorname{op}} \rightarrow \operatorname{End}_{B'}(eA).$$

is an isomorphism too, which implies the Schur functor F is fully faithful on projectives. This means

$$(3.24) \quad \operatorname{Hom}_A(P^\lambda, P^\mu) \simeq \operatorname{Hom}_{B'}(eP^\lambda, eP^\mu) \quad \text{for all } \lambda, \mu \in \Lambda.$$

In particular, $Y^\lambda := eP^\lambda$ is an indecomposable B' -module for all $\lambda \in \Lambda$. Note that $\{F(L^\lambda) \mid \lambda \in \Lambda_0\}$ is a complete list of non-isomorphic simple B' -modules and Y^λ is the projective cover of simple B' -module $F(L^\lambda)$ for all $\lambda \in \Lambda_0$. For $\lambda, \mu \in \Lambda_0$ such that L^λ and L^μ are in the same block of A , there exists a sequence $\lambda = \gamma_1, \gamma_2, \dots, \gamma_l = \mu$ of elements in Λ such that $\operatorname{Hom}_A(P^{\gamma_j}, P^{\gamma_{j+1}}) \neq 0$ or $\operatorname{Hom}_A(P^{\gamma_{j+1}}, P^{\gamma_j}) \neq 0$ for all $1 \leq j \leq l-1$ from the discussion before this proposition. Since Y^γ are indecomposable B' -modules for all $\gamma \in \Lambda$, Y^{γ_j} and $Y^{\gamma_{j+1}}$ in the same block of B' for all $1 \leq j \leq l-1$ by (3.24) and therefore $F(L^\lambda)$ and $F(L^\mu)$ in the same block of B' . This implies $n \leq m$ and hence $m = n$. Also we obtain that the functor F induces a bijection between the indecomposable blocks of A and B' .

Let $\lambda, \mu \in \Lambda_0$ such that P^λ and P^μ belong to the same block of A . Now we are going to show that $d_\lambda = d_\mu$. From above, the projective modules Y^λ and Y^μ belong to the same block B' . From the discussion before this proposition, there exists a sequence $\lambda = \gamma_1, \gamma_2, \dots, \gamma_l = \mu$ of elements in Λ_0 such that $\operatorname{Hom}_{B'}(Y^{\gamma_j}, Y^{\gamma_{j+1}}) \neq 0$ or $\operatorname{Hom}_{B'}(Y^{\gamma_{j+1}}, Y^{\gamma_j}) \neq 0$ for all $1 \leq j \leq l-1$ since $\{Y^\lambda \mid \lambda \in$

$\Lambda_0\}$ is a complete set of non-isomorphic indecomposable projective B^0 -modules. By (3.24), we have $\text{Hom}_A(P^{\gamma_j}, P^{\gamma_{j+1}}) \neq 0$ or $\text{Hom}_A(P^{\gamma_{j+1}}, P^{\gamma_j}) \neq 0$ for all $1 \leq j \leq l-1$. To show that $d_\lambda = d_\mu$, it suffices to show that $d_{\lambda_j} = d_{\lambda_{j-1}}$ for all $1 \leq j \leq l-1$. Therefore it is enough to show $d_\lambda = d_\mu$ when $\text{Hom}_A(P^\mu, P^\lambda) \neq 0$ and $\lambda, \mu \in \Lambda_0$.

In the case that (A, \star) is a \mathbb{Z} -graded cellular algebra in the sense of [7], by [7, Theorem 2.17], we have

$$c_{\lambda, \mu}(v) = \sum_{\gamma} [C^\gamma : L^\lambda]_v [C^\gamma : L^\mu]_v = c_{\mu, \lambda}(v), \quad \text{for all } \mu, \lambda \in \Lambda.$$

Recall that $[M : L^\beta]_v$ is defined in (3.21) to be the graded multiplicities of L^β in the graded module M .

Now we assume that $(A\text{-gmod}, \otimes)$ is a \mathbb{Z} -graded highest weight category with a duality functor \otimes induced from \star . Then we have (3.15) and (3.16). By [14, Corollary 2.16], we have that

$$(P^\lambda : \Delta^\gamma)_v = [\Delta^\gamma : L^\lambda]_v, \quad \forall \gamma, \lambda \in \Lambda,$$

where $(P^\lambda : \Delta^\gamma)_v$ is defined to be the graded filtration multiplicities of Δ^γ in P^λ . It follows that

$$c_{\lambda, \mu}(v) = \sum_{\gamma} (P^\lambda : \Delta^\gamma)_v [\Delta^\gamma : L^\mu]_v = \sum_{\gamma} [\Delta^\gamma : L^\lambda]_v [\Delta^\gamma : L^\mu]_v = c_{\mu, \lambda}(v), \quad \text{for all } \mu, \lambda \in \Lambda.$$

Therefore, in both cases, we have $c_{\lambda, \mu}(v) = c_{\mu, \lambda}(v)$ for all $\mu, \lambda \in \Lambda$. On the other hand, we have by (3.18) that

$$\begin{aligned} c_{\lambda, \mu}(v) &= \dim_v \text{Hom}_A(P^\mu, P^\lambda) = \dim_v \text{Hom}_A((P^\lambda)^\otimes, (P^\mu)^\otimes) = \dim_v \text{Hom}_A(P^\lambda \langle -d_\lambda \rangle, P^\mu \langle -d_\mu \rangle) \\ &= v^{-d_\mu + d_\lambda} \dim_v \text{Hom}_A(P^\lambda, P^\mu) = v^{-d_\mu + d_\lambda} c_{\mu, \lambda}(v) = v^{-d_\mu + d_\lambda} c_{\lambda, \mu}(v), \end{aligned}$$

By assumption that $c_{\lambda, \mu}(v) \neq 0$, thus we can deduce that $v^{d_\mu - d_\lambda} = 1$ and hence $d_\mu = d_\lambda$ as required. \square

4. PARABOLIC BGG CATEGORY \mathcal{O} AND THE PROOF OF THEOREM 1.2

In this section, we shall study the endomorphism algebra of any projective-injective module in parabolic BGG category \mathcal{O}^p over the field \mathbb{C} of complex numbers. After recalling some preliminarily results on parabolic BGG category \mathcal{O}^p , we then give a proof of Theorem 1.2. One of the key ingredient in the proof of Theorem 1.2 is Proposition 4.25.

Let \mathfrak{g} be a complex semisimple Lie algebra with a fixed Borel subalgebra \mathfrak{b} containing the Cartan subalgebra \mathfrak{h} and let \mathcal{O} denote the corresponding Bernstein-Gelfand-Gelfand (BGG) category [9]. Let Φ be the root system of \mathfrak{g} relative to \mathfrak{h} and Δ the set of simple roots in Φ corresponding to \mathfrak{b} . Let W be the Weyl group of \mathfrak{g} attached to the root system Φ . An element $\lambda \in \mathfrak{h}^*$ is called a weight of \mathfrak{g} . For any weight $\lambda \in \mathfrak{h}^*$ and $w \in W$, we define $w \cdot \lambda := w(\lambda + \rho) - \rho$, where ρ is the half-sum of positive roots in Φ . For any $\lambda \in \mathfrak{h}^*$, let $L(\lambda) \in \text{Obj}(\mathcal{O})$ denote the irreducible highest weight module with highest weight λ . A weight $\lambda \in \mathfrak{h}^*$ is said to be integral (resp. dominant) if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ (resp. $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$) for any $\alpha \in \Delta$. A dominant integral weight λ is called regular if $\langle \lambda + \rho, \alpha^\vee \rangle > 0$ for any $\alpha \in \Delta$, otherwise λ is called singular. Let Λ denote the set of integral weights of \mathfrak{g} .

Let $I \subset \Delta$ be a subset of Δ which defines a subroot system $\Phi_I \subset \Phi$ with positive roots Φ_I^+ , negative roots Φ_I^- and Weyl group W_I generated by all s_α with $\alpha \in I$. Associated with the root system Φ_I we have the standard parabolic subalgebra $\mathfrak{p} := \mathfrak{p}_I \supseteq \mathfrak{b}$ which has a Levi decomposition $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$, where $\mathfrak{l}_I := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$ is the Levi subalgebra of \mathfrak{p} and $\mathfrak{u}_I := \bigoplus_{\alpha \in \Phi \setminus \Phi_I} \mathfrak{g}_\alpha$ is the nilradical of \mathfrak{p} . We define $\Lambda_{\mathfrak{p}}^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0, \text{ for all } \alpha \in I\}$. Let \mathcal{O}^p denote the corresponding parabolic BGG category [18]. For each $\lambda \in \Lambda$, $L(\lambda)$ lies in \mathcal{O}^p if and only if $\lambda \in \Lambda_{\mathfrak{p}}^+$. For each $\lambda \in \Lambda$, let \mathcal{O}_λ denote the subcategory of \mathcal{O} whose composition factors are all of the form $L(w \cdot \lambda)$ for some $w \in W$ and let $\mathcal{O}_\lambda^p := \mathcal{O}_\lambda \cap \mathcal{O}^p$. For $\lambda \in \Lambda_{\mathfrak{p}}^+$, let $\Delta(\lambda)$ denote the parabolic Verma module with highest weight λ and $P(\lambda)$ denote the projective cover of $L(\lambda)$ in \mathcal{O}^p .

For any integral dominant weight ψ , let $W_\psi := \{w \in W \mid w \cdot \psi = \psi\}$ denote the stabiliser of ψ in the Weyl group W and let \dot{W}^ψ denote the set of maximal length left coset representatives of W_ψ in W and let

$$\dot{W}^\psi := \{w \in W^\psi \mid w \cdot \psi \in \Lambda_{\mathfrak{p}}^+\}.$$

The simple modules $L(\mu)$ in \mathcal{O}_ψ^p are parametrized by $\mu \in \dot{W}^\psi \cdot \psi$. Let $P(\mu)$ denote the projective cover of $L(\mu)$ in \mathcal{O}_ψ^p for all $\mu \in \dot{W}^\psi \cdot \psi$. Let $(-)^*$ denote the dual functor on the BGG category \mathcal{O} . The dual functor $(-)^*$ on \mathcal{O} descending to the parabolic BGG category \mathcal{O}^p is also denoted by $(-)^*$ (see, for example, [9, 3.2], in which $(-)^*$ is denoted by “ $(-)^{\vee}$ ”). Recall that a module M is called self-dual if

M^* is isomorphic to M . Note that $L(\mu)^* \cong L(\mu)$ for all $\mu \in \Lambda_{\mathfrak{p}}^+$. That is, every simple module in $\mathcal{O}^{\mathfrak{p}}$ is self-dual.

Let λ denote a fixed dominant integral weight. Let $T_0^\lambda : \mathcal{O}_0 \rightarrow \mathcal{O}_\lambda$ and $T_\lambda^0 : \mathcal{O}_\lambda \rightarrow \mathcal{O}_0$ be the two translation functors defined by $T_0^\lambda(-) := \text{pr}_\lambda(E \otimes -)$ and $T_\lambda^0(-) := \text{pr}_0(F \otimes -)$ [13] (see also [9, Chapter 7], [2]), where E is a finite-dimensional irreducible \mathfrak{g} -module with extremal weight λ and pr_λ is the projection from \mathcal{O} onto the block \mathcal{O}_λ ; and F is a finite-dimensional irreducible \mathfrak{g} -module with extremal weight $-\lambda$ and pr_0 is the projection from \mathcal{O} onto the block \mathcal{O}_0 . It is well-known that the functors T_0^λ and T_λ^0 induce the functors $T_0^\lambda : \mathcal{O}_0^{\mathfrak{p}} \rightarrow \mathcal{O}_\lambda^{\mathfrak{p}}$ and $T_\lambda^0 : \mathcal{O}_\lambda^{\mathfrak{p}} \rightarrow \mathcal{O}_0^{\mathfrak{p}}$ on the parabolic categories. Here and after, the functors T_0^λ and T_λ^0 stand for the functors on the parabolic categories. The following lemma is a collection of some well-known results of the translation functors on the parabolic categories (see, for example, [2, Lemma 2.5, 2.6], [9, Pages 130, 140, 143, 186, 192]).

4.1. Lemma. *For a dominant integral weight λ , we have the following:*

- (i) $T_0^\lambda : \mathcal{O}_0^{\mathfrak{p}} \rightarrow \mathcal{O}_\lambda^{\mathfrak{p}}$ and $T_\lambda^0 : \mathcal{O}_\lambda^{\mathfrak{p}} \rightarrow \mathcal{O}_0^{\mathfrak{p}}$ are exact functors adjoint to each other. Moreover, for any $M, N \in \mathcal{O}_\lambda^{\mathfrak{p}}$ and $f \in \text{Hom}_{\mathcal{O}}(M, N)$, we have

$$T_0^\lambda T_\lambda^0(M) \cong \underbrace{M \oplus \cdots \oplus M}_{|W_\lambda| \text{ copies}}, \quad T_0^\lambda T_\lambda^0(f) \cong \underbrace{f \oplus \cdots \oplus f}_{|W_\lambda| \text{ copies}}.$$

- (ii) For $w \in W$ with $w \cdot 0 \in \Lambda_{\mathfrak{p}}^+$, we have

$$T_0^\lambda(L(w \cdot 0)) \cong \begin{cases} L(w \cdot \lambda), & \text{if } w \in \dot{W}^\lambda; \\ 0, & \text{if } w \notin \dot{W}^\lambda. \end{cases}$$

- (iii) T_0^λ and T_λ^0 send projectives to projectives. Moreover, if $w \in \dot{W}^\lambda$, then $T_\lambda^0(P(w \cdot \lambda)) = P(w \cdot 0)$.

- (iv) T_0^λ and T_λ^0 commute with the dual functor $(-)^*$.

4.2. Lemma. *For $w \in \dot{W}^\lambda$, $T_\lambda^0(L(w \cdot \lambda))$ is self-dual with a simple head isomorphic to $L(w \cdot 0)$ and a simple socle isomorphic to $L(w \cdot 0)$. Moreover, we have*

$$[T_\lambda^0(L(w \cdot \lambda)) : L(y \cdot 0)] = \delta_{y,w} |W_\lambda|, \quad \text{for any } y \in \dot{W}^\lambda.$$

Proof. By Lemma 4.1 (i) and (ii), for any $y \in W$ with $y \cdot 0 \in \Lambda_{\mathfrak{p}}^+$, we have,

$$\begin{aligned} \text{Hom}_{\mathcal{O}}(T_\lambda^0(L(w \cdot \lambda)), L(y \cdot 0)) &\cong \text{Hom}_{\mathcal{O}}(L(w \cdot \lambda), T_0^\lambda(L(y \cdot 0))) \\ &\cong \begin{cases} \text{Hom}_{\mathcal{O}}(L(w \cdot \lambda), L(y \cdot \lambda)) \cong \delta_{y,w} \mathbb{C}, & \text{if } y \in \dot{W}^\lambda; \\ 0, & \text{if } y \notin \dot{W}^\lambda, \end{cases} \\ &= \delta_{y,w} \mathbb{C}. \end{aligned}$$

This proves that $T_\lambda^0(L(w \cdot \lambda))$ has a simple head isomorphic to $L(w \cdot 0)$. Since $(T_\lambda^0(L(w \cdot \lambda)))^* \cong T_\lambda^0(L(w \cdot \lambda)^*) \cong T_\lambda^0(L(w \cdot \lambda))$ and $T_\lambda^0(L(w \cdot \lambda))$ has a simple head isomorphic to $L(w \cdot 0)$, $T_\lambda^0(L(w \cdot \lambda))$ is self-dual with a simple socle isomorphic to $L(w \cdot 0)$.

Finally, by Lemma 4.1 (i) and (ii), for any $y \in \dot{W}^\lambda$ satisfying $[T_\lambda^0(L(w \cdot \lambda)) : L(y \cdot 0)] \neq 0$, we have $L(y \cdot \lambda) \cong T_0^\lambda(L(y \cdot 0))$ and hence

$$[T_0^\lambda(T_\lambda^0(L(w \cdot \lambda)) : L(y \cdot \lambda)] = [T_\lambda^0(L(w \cdot \lambda) : L(y \cdot 0)].$$

By Lemma 4.1 (i) again, we get that $y = w$ and $[T_\lambda^0(L(w \cdot \lambda)) : L(w \cdot 0)] = |W_\lambda|$. \square

4.3. Remark. A graded version of Lemma 4.2 is given in Lemma 4.18 below.

4.4. Definition. For $\gamma \in \Lambda_{\mathfrak{p}}^+$, λ is said to be *socular* if $L(\gamma)$ lies in the socle of a parabolic Verma module $\Delta(\mu)$ for some $\mu \in \Lambda_{\mathfrak{p}}^+$.

4.5. Lemma. ([10, Addendum]) *For $\gamma \in \Lambda_{\mathfrak{p}}^+$, $P(\gamma)$ is injective if and only if γ is socular. In this case, $P(\gamma)$ is a tilting module and in particular it is self-dual.*

Lemma 4.5 implies that each block of $\mathcal{O}_\psi^{\mathfrak{p}}$ contains a projective-injective module for any dominant integral weight ψ .

4.6. Lemma. *Let $w \in \dot{W}^\lambda$. If $w \cdot \lambda$ is socular, then $w \cdot 0$ is socular too.*

Proof. By assumption that $w \cdot \lambda$ is socular, it follows that $P(w \cdot \lambda)$ is self-dual and hence $P(w \cdot 0) \cong T_\lambda^0(P(w \cdot \lambda))$ is also self-dual. Therefore $w \cdot 0$ is a socular weight. \square

4.7. Definition. For a dominant integral weight ψ , we set

$$(4.8) \quad \Lambda_0^\psi := \{\mu \in \check{W}^\psi \cdot \psi \mid \mu \text{ is socular}\} \quad \text{and} \quad \check{W}^\psi := \{w \in \check{W}^\psi \mid w \cdot \psi \text{ is socular}\}.$$

It is clear that the map $w \mapsto w \cdot \psi$ defines a bijection between \check{W}^ψ and Λ_0^ψ . Note that $\check{W}^0 = \{w \in W \mid w \cdot 0 \in \Lambda_p^+ \text{ and } w \cdot 0 \text{ is socular}\}$. It follows from Lemma 4.6 that

$$(4.9) \quad \check{W}^\lambda \subset \check{W}^0.$$

4.10. Definition. For a dominant integral weight ψ , we define the basic algebra of the category \mathcal{O}_ψ^p by

$$A^p(\psi) := \left(\text{End}_{\mathcal{O}^p} \left(\bigoplus_{\mu \in \check{W}^\psi} P(\mu) \right) \right)^{\text{op}}.$$

The functor $\mathcal{F}^\psi := \text{Hom}_{\mathcal{O}^p}(\bigoplus_{\mu \in \check{W}^\psi} P(\mu), -)$ gives the equivalence of categories

$$(4.11) \quad \mathcal{F}^\psi : \mathcal{O}_\psi^p \cong A^p(\psi)\text{-mod}.$$

4.12. Definition. Let ψ be a dominant integral weight. For each $\mu \in \check{W}^\psi \cdot \psi$, we define

$$L_b^\mu := \mathcal{F}^\psi(L(\mu)), \quad \Delta_b^\mu := \mathcal{F}^\psi(\Delta(\mu)), \quad P_b^\mu := \mathcal{F}^\psi(P(\mu)).$$

By [3, Theorem 1.1.3] and [2, Theorem 1.1] (see also [19, 8.1, 8.2, 8.4]), $A^p(\psi)$ can be endowed with a Koszul \mathbb{Z} -grading. Let $\tilde{A}^p(\psi)$ denote the algebra $A^p(\psi)$ equipped with the Koszul \mathbb{Z} -grading. Hence $\tilde{A}^p(\psi)$ is a positively graded algebra with $\tilde{A}^p(\psi)_0$ spanned by orthogonal idempotents. In particular, simple and projective $A^p(\psi)$ -modules have graded lifts and every simple $\tilde{A}^p(\psi)$ -module is one-dimensional concentrated in a fixed degree. For each $\mu \in \check{W}^\psi \cdot \psi$, let \tilde{L}_b^μ be the graded lift of simple module L_b^μ concentrated in degree 0 and let

$$(4.13) \quad \tilde{P}_b^\mu = \sum_{j=0}^{d_\mu} (\tilde{P}_b^\mu)_j \quad \text{with } (\tilde{P}_b^\mu)_{d_\mu} \neq 0$$

be the graded lift of the projective module P_b^μ such that the natural projection from \tilde{P}_b^μ to \tilde{L}_b^μ is a homogenous homomorphism of degree 0.

By [3, Theorem 1.1.3, Proposition 2.4.1] and [2, Theorem 1.1], $\tilde{A}^p(\lambda)$ is Koszul and each indecomposable projective-injective module is rigid in the sense that both the radical filtration and the socle filtration of any indecomposable projective-injective module coincide with its grading filtration up to shift. In particular, the graded length of any indecomposable projective-injective module is the same as its Loewy length. By Corollary 3.12, the following proposition obtained by Coulembier and Mazorchuk provides a necessary condition for the existence of the structure of symmetric algebra on the endomorphism algebra B_0^p (see Definition 4.20 below).

4.14. Proposition. [15] (cf. [17, Theorem 5.2, Remark 5.3]) *Every indecomposable projective-injective module of $\tilde{A}^p(\lambda)$ has the same graded length and hence the same Loewy length.*

4.15. Remark. Note that a (weaker) block version of the above proposition can also be deduced as a consequence of Proposition 3.22. We sketch a proof as follows. It is well known that $\tilde{A}^p(\lambda)\text{-gmod}$ is a \mathbb{Z} -graded highest weight category with a duality functor. The double centralizer property in the assumptions of Proposition 3.22 also hold by [20, Theorem 10.1] and [17, Examples 2.7 (2)]. Now all the assumptions of Proposition 3.22 are satisfied and hence the block version of the proposition above follows.

Using the categorical equivalences in (4.11), we shall simply regard the functors T_0^λ, T_λ^0 as the functors between $A^p(0)\text{-mod}$ and $A^p(\lambda)\text{-mod}$ without further explanation. Applying [2, Page 147, the 4th paragraph], both the functors T_λ^0 and T_0^λ have graded lifts, i.e., we have graded translation functors:

$$\tilde{T}_\lambda^0 : \tilde{A}^p(\lambda)\text{-gmod} \rightarrow \tilde{A}^p(0)\text{-gmod} \quad \text{and} \quad \tilde{T}_0^\lambda : \tilde{A}^p(0)\text{-gmod} \rightarrow \tilde{A}^p(\lambda)\text{-gmod}.$$

Moreover, \tilde{T}_0^λ is a right adjoint functor of \tilde{T}_λ^0 and

$$(4.16) \quad \tilde{T}_0^\lambda(\tilde{L}_b^{w \cdot \lambda}) \simeq \tilde{L}_b^{w \cdot \lambda} \quad \text{for all } w \in \check{W}^\lambda.$$

Recall that $M \simeq N$ defined in Section 3 denotes that there is a homogeneous isomorphism of degree 0 between graded modules M and N . Therefore we have, for all $w, y \in \check{W}^\lambda$ and $k \in \mathbb{Z}$,

$$\text{hom}_{\tilde{A}^p(0)}(\tilde{T}_\lambda^0(\tilde{P}_b^{w \cdot \lambda}), \tilde{L}_b^{y \cdot 0}(k)) \cong \text{hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w \cdot \lambda}, \tilde{T}_0^\lambda(\tilde{L}_b^{y \cdot 0}(k))) \cong \text{hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w \cdot \lambda}, \tilde{L}_b^{y \cdot \lambda}(k)) \cong \delta_{k,0} \delta_{w,y} \mathbb{C}$$

and hence

$$(4.17) \quad \tilde{T}_\lambda^0(\tilde{P}_b^{w \cdot \lambda}) \simeq \tilde{P}_b^{w \cdot 0} \quad \text{for all } w \in \check{W}^\lambda.$$

The following lemma is a graded version of Lemma 4.2. Note that d_μ is defined in (4.13).

4.18. Lemma. *For $w \in \check{W}^\lambda$, we have*

- (i) $\text{head}(\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})) \simeq \tilde{L}_b^{w \cdot 0}$, $\text{soc}(\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})) \simeq \tilde{L}_b^{w \cdot 0} \langle d_{w \cdot 0} - d_{w \cdot \lambda} \rangle$;
- (ii) $\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})$ is rigid, and both the radical filtration and the socle filtration of $\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})$ coincide with its grading filtration (up to a grading shift), and the graded length of $\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})$ is $d_{w \cdot 0} - d_{w \cdot \lambda}$. In particular, $d_{w \cdot 0} \geq d_{w \cdot \lambda}$;
- (iii) $[\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda}) : \tilde{L}_b^{w \cdot 0} \langle k \rangle] \neq 0$ only if $0 \leq k \leq d_{w \cdot 0} - d_{w \cdot \lambda}$. Moreover,

$$[\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda}) : \tilde{L}_b^{w \cdot 0}] = 1 = [\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda}) : \tilde{L}_b^{w \cdot 0} \langle d_{w \cdot 0} - d_{w \cdot \lambda} \rangle].$$

Proof. Applying Lemma 4.2, we see that $\text{head}(T_\lambda^0(L_b^{w \cdot \lambda})) \cong L_b^{w \cdot 0}$ and $\text{soc}(T_\lambda^0(L_b^{w \cdot \lambda})) \cong L_b^{w \cdot 0}$. Now the degree 0 surjection $\tilde{P}_b^{w \cdot \lambda} \twoheadrightarrow \tilde{L}_b^{w \cdot \lambda}$ naturally induces a degree 0 surjection $\tilde{P}_b^{w \cdot 0} \simeq \tilde{T}_\lambda^0(\tilde{P}_b^{w \cdot \lambda}) \twoheadrightarrow \tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})$, which implies that $\text{head}(\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})) \simeq \tilde{L}_b^{w \cdot 0}$. Similarly, the degree 0 embedding $\tilde{L}_b^{w \cdot \lambda} \hookrightarrow \tilde{P}_b^{w \cdot \lambda}$ naturally induces a degree 0 embedding

$$\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda}) \langle d_{w \cdot \lambda} \rangle = \tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda} \langle d_{w \cdot \lambda} \rangle) \hookrightarrow \tilde{T}_\lambda^0(\tilde{P}_b^{w \cdot \lambda}) \simeq \tilde{P}_b^{w \cdot 0}.$$

Since $\text{soc}(\tilde{P}_b^{w \cdot 0}) = \tilde{L}_b^{w \cdot 0} \langle d_{w \cdot 0} \rangle$, it follows that $\text{soc}(\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})) \simeq \tilde{L}_b^{w \cdot 0} \langle d_{w \cdot 0} - d_{w \cdot \lambda} \rangle$ as required. This proves (i).

From above, $\tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda})$ has a unique simple head as well as a unique simple socle, we see that (ii) follows from [3, Proposition 2.4.1], the fact that $\tilde{A}^p(0)$ is Koszul ([3, Theorem 1.1.3]) and (i). Finally, (iii) follows from (ii) and (i). \square

The following Lemma is a consequence of Lemma 4.18, Lemma 4.1(i), (ii) and (4.16).

4.19. Lemma. *For $w \in \check{W}^\lambda$, there exist a sequence of integers $0 = a_1^w < a_2^w \leq \dots \leq a_{|W_\lambda|-1}^w < a_{|W_\lambda|}^w = d_{w \cdot 0} - d_{w \cdot \lambda}$ such that*

$$\tilde{T}_0^\lambda \tilde{T}_\lambda^0(\tilde{L}_b^{w \cdot \lambda}) \simeq \bigoplus_{j=1}^{|W_\lambda|} \tilde{L}_b^{w \cdot \lambda} \langle a_j^w \rangle \quad \text{and} \quad \tilde{T}_0^\lambda(\tilde{P}_b^{w \cdot 0}) \simeq \tilde{T}_0^\lambda \tilde{T}_\lambda^0(\tilde{P}_b^{w \cdot \lambda}) \simeq \bigoplus_{j=1}^{|W_\lambda|} \tilde{P}_b^{w \cdot \lambda} \langle a_j^w \rangle.$$

4.20. Definition. For a dominant integral weight ψ , the endomorphism algebra of the basic projective-injective module of \mathcal{O}_ψ^p is denoted by

$$B_\psi^p := \text{End}_{\tilde{A}^p(\psi)}(\bigoplus_{w \in \check{W}^\psi} \tilde{P}_b^{w \cdot \psi}).$$

It is clear that B_ψ^p is a positively graded \mathbb{C} -algebra.

Since B_0^p is a symmetric algebra over \mathbb{C} [17, Theorem 4.6], there are an admissible basis \mathcal{B} and the canonical symmetrizing form tr attached to \mathcal{B} on B_0^p by Corollary 3.12 and Proposition 3.9. For each $w \in \check{W}^0$, there is a unique endomorphism $\theta_{w \cdot 0} \in \mathcal{B} \cap \text{End}_{\tilde{A}^p(0)}(\tilde{P}_b^{w \cdot 0})_{d_{w \cdot 0}}$. Note that $\theta_{w \cdot 0}(\tilde{P}_b^{w \cdot 0}) = \text{soc}(\tilde{P}_b^{w \cdot 0}) \simeq \tilde{L}_b^{w \cdot 0} \langle d_{w \cdot 0} \rangle$. We may assume that $\text{tr}(\theta_{w \cdot 0}) = 1$ for all $w \in \check{W}^0$.

Recall that $\check{W}^\lambda \subset \check{W}^0$ by (4.9). From now on, we will fix an isomorphism $\tilde{T}_0^\lambda(\tilde{P}_b^{w \cdot 0}) \simeq \bigoplus_{j=1}^{|W_\lambda|} \tilde{P}_j^{w \cdot \lambda}$ for each $w \in \check{W}^\lambda$ obtained in Lemma 4.19, where $\tilde{P}_j^{w \cdot \lambda} := \tilde{P}_b^{w \cdot \lambda} \langle a_j^w \rangle$ for all $1 \leq j \leq |W_\lambda|$. For $w \in \check{W}^\lambda$, we will use the matrix notations to write any element in $\text{Hom}_{\tilde{A}^p(\lambda)}(\bigoplus_{j=1}^{|W_\lambda|} \tilde{P}_j^{w \cdot \lambda}, \bigoplus_{j=1}^{|W_\lambda|} \tilde{P}_j^{w \cdot \lambda})$ in the form of

$$\sum_{1 \leq i, j \leq |W_\lambda|} h_{ji}, \quad \text{where } h_{ji} \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_j^{w \cdot \lambda}, \tilde{P}_i^{w \cdot \lambda}).$$

Therefore any element in $\text{End}_{\tilde{A}^p(\lambda)}(\bigoplus_{j=1}^{|W_\lambda|} \tilde{P}_j^{w \cdot \lambda})$ can be written in the form

$$(4.21) \quad \sum_{1 \leq i, j \leq |W_\lambda|} g_{ji} [a_i^w] \pi_{ji}^w,$$

where $g_{ji} \in \text{End}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w \cdot \lambda})$ and $\pi_{ji}^w := \text{id}_{\tilde{P}_b^{w \cdot \lambda}}[a_j, a_i]$, where $\text{id}_{\tilde{P}_b^{w \cdot \lambda}}$ is the identity map on $\tilde{P}_b^{w \cdot \lambda}$, and $f[j, i]$ and $f[i]$ are defined in (3.2) and (3.3) for a homomorphism f between graded modules. In other

words, π_{ji}^w is a homogeneous homomorphism from $\tilde{P}_j^{w,\lambda}$ onto $\tilde{P}_i^{w,\lambda}$ of degree $a_i^w - a_j^w$ such that $\underline{\text{For}}(\pi_{ji}^w)$ is the identity map on $P_b^{w,\lambda}$ for all i, j .

Since $\deg(\theta_{w,0}) = d_{w,0}$ for $w \in \check{W}^\lambda$, we have $\deg(\tilde{T}_0^\lambda(\theta_{w,0})) = d_{w,0}$ and hence

$$(4.22) \quad \tilde{T}_0^\lambda(\theta_{w,0}) = \theta_{w,\lambda}[d_{w,0} - d_{w,\lambda}]\pi_{1,|W_\lambda|}^w$$

by Lemma 4.19 and the fact that $0 \leq \deg f \leq d_{w,\lambda}$ for any homogeneous map $f \in \text{End}_{\tilde{A}^p(\lambda)}(P_b^{w,\lambda})$, where $\theta_{w,\lambda} \in \text{End}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w,\lambda})_{d_{w,\lambda}}$ satisfies $\theta_{w,\lambda}(\tilde{P}_b^{w,\lambda}) = \text{soc}(\tilde{P}_b^{w,\lambda})$. Recall that $f[k]$ is defined in (3.2) and (3.3). The $\theta_{w,\lambda}$ s defined from above play crucial roles for the rest of the paper.

Since B_0^p is a symmetric algebra over \mathbb{C} [17, Theorem 4.6], for every $w \in \check{W}^\lambda$ there is a homogeneous homomorphism $\bar{\theta}_w \in \text{End}_{\tilde{A}^p(0)}(\tilde{P}_b^{w,0})_{d_{w,0}-d_{w,\lambda}}$ by Proposition 3.10 such that

$$(4.23) \quad \bar{\theta}_w \tilde{T}_\lambda^0(\theta_{w,\lambda}) = \theta_{w,0} = \tilde{T}_\lambda^0(\theta_{w,\lambda}) \bar{\theta}_w.$$

Let

$$\bar{\theta} := \sum_{w \in \check{W}^\lambda} \bar{\theta}_w.$$

4.24. Lemma. *For each $w \in \check{W}^\lambda$, we can write*

$$\tilde{T}_0^\lambda(\bar{\theta}_w) = \sum_{1 \leq i, j \leq |W_\lambda|} q_{ji}^w[a_i^w] \pi_{ji}^w,$$

where $q_{ji}^w \in \text{End}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w,\lambda})$ such that $\deg(q_{ji}^w) > 0$ or $q_{ji}^w = 0$ for all $(j, i) \neq (1, |W_\lambda|)$ and $q_{1,|W_\lambda|}^w$ is the identity map.

Proof. For each $w \in \check{W}^\lambda$, we can write

$$\tilde{T}_0^\lambda(\bar{\theta}_w) = \sum_{1 \leq i, j \leq |W_\lambda|} q_{ji}^w[a_i^w] \pi_{ji}^w,$$

where $q_{ji}^w \in \text{End}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w,\lambda})$ for all i, j by (4.21). Note that $\tilde{T}_0^\lambda \tilde{T}_\lambda^0(\theta_{w,\lambda}) = \sum_{1 \leq j \leq |W_\lambda|} \theta_{w,\lambda}[a_j^w]$ by Lemma 4.1 and Lemma 4.19. By (4.22) and (4.23), we have

$$\begin{aligned} \theta_{w,\lambda}[d_{w,0} - d_{w,\lambda}]\pi_{1,|W_\lambda|}^w &= \tilde{T}_0^\lambda(\theta_{w,0}) \\ &= \tilde{T}_0^\lambda \tilde{T}_\lambda^0(\theta_{w,\lambda}) \tilde{T}_0^\lambda(\bar{\theta}_w) \\ &= \sum_{1 \leq j \leq |W_\lambda|} \theta_{w,\lambda}[a_j^w] \sum_{1 \leq i, j \leq |W_\lambda|} q_{ji}^w[a_i^w] \pi_{ji}^w \\ &= \sum_{1 \leq i, j \leq |W_\lambda|} \theta_{w,\lambda}[a_i^w] q_{ji}^w[a_i^w] \pi_{ji}^w \\ &= \sum_{1 \leq i, j \leq |W_\lambda|} (\theta_{w,\lambda} q_{ji}^w)[a_i^w] \pi_{ji}^w. \end{aligned}$$

Therefore $(\theta_{w,\lambda} q_{ji}^w)[a_i^w] = 0$ for all $(j, i) \neq (1, |W_\lambda|)$ and $(\theta_{w,\lambda} q_{ji}^w)[a_i^w] = \theta_{w,\lambda}[d_{w,0} - d_{w,\lambda}]$ for $(j, i) = (1, |W_\lambda|)$. Hence $\deg(q_{ji}^w) > 0$ or $q_{ji}^w = 0$ for all $(j, i) \neq (1, |W_\lambda|)$ and $q_{1,|W_\lambda|}^w$ is the identity map. \square

The following proposition will play a key role in the proof of Theorem 1.2.

4.25. Proposition. *For $w, y \in \check{W}^\lambda$, we have*

$$(4.26) \quad \tilde{T}_\lambda^0(f) \tilde{T}_\lambda^0(h) \bar{\theta}_y = \tilde{T}_\lambda^0(f) \bar{\theta}_w \tilde{T}_\lambda^0(h),$$

for all $f \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w,\lambda}, \tilde{P}_b^{y,\lambda})_k$, $h \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{y,\lambda}, \tilde{P}_b^{w,\lambda})_{d_{y,\lambda}-k}$ and $k \in \mathbb{Z}$.

Moreover, if $\text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w,\lambda}, \tilde{P}_b^{y,\lambda}) \neq 0$, then $d_{w,\lambda} = d_{y,\lambda}$.

Proof. We first compute $\tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y)$ and $\tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h))$ and show they are equal. By Lemma 4.24, we have $\tilde{T}_0^\lambda(\bar{\theta}_y) = \sum_{1 \leq i, j \leq |W_\lambda|} q_{ji}^y[a_i^y]\pi_{ji}^y$ and

$$\begin{aligned} \tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y) &= \tilde{T}_0^\lambda\tilde{T}_\lambda^0(fh)\tilde{T}_0^\lambda(\bar{\theta}_y) \\ &= \left(\sum_{1 \leq i \leq |W_\lambda|} (fh)[a_i^y]\right) \left(\sum_{1 \leq i, j \leq |W_\lambda|} q_{ji}^y[a_i^y]\pi_{ji}^y\right) \\ &= \sum_{1 \leq i, j \leq |W_\lambda|} (fhq_{ji}^y)[a_i^y]\pi_{ji}^y \\ &= (fhq_{1, |W_\lambda|}^y)[a_{|W_\lambda|}^y]\pi_{1, |W_\lambda|}^y \\ &= (fh)[a_{|W_\lambda|}^y]\pi_{1, |W_\lambda|}^y. \end{aligned}$$

The second equality follows from Lemma 4.1(i) and Lemma 4.19, the fourth equality follows from $fhq_{ji}^y = 0$ for all $(j, i) \neq (1, |W_\lambda|)$ because $\deg(fhq_{ji}^y) > d_{y, \lambda}$ for all $(j, i) \neq (1, |W_\lambda|)$ by Lemma 4.24 and the assumptions, the final equality follows from $q_{1, |W_\lambda|}^y$ is the identity map by Lemma 4.24. Similarly,

$$\begin{aligned} \tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h)) &= \left(\sum_{1 \leq i \leq |W_\lambda|} f[a_i^w, a_i^y]\right) \left(\sum_{1 \leq i, j \leq |W_\lambda|} q_{ji}^w[a_i^w]\pi_{ji}^w\right) \left(\sum_{1 \leq j \leq |W_\lambda|} h[a_j^y, a_j^w]\right) \\ &= \sum_{1 \leq i, j \leq |W_\lambda|} f[a_i^w, a_i^y]q_{ji}^w[a_i^w]\pi_{ji}^w h[a_j^y, a_j^w] \\ &= f[a_{|W_\lambda|}^w, a_{|W_\lambda|}^y]q_{1, |W_\lambda|}^w[a_{|W_\lambda|}^w]\pi_{1, |W_\lambda|}^w h[0, 0] \\ &= f[a_{|W_\lambda|}^w, a_{|W_\lambda|}^y]\pi_{1, |W_\lambda|}^w h[0, 0] \\ &= (fh)[a_{|W_\lambda|}^y]\pi_{1, |W_\lambda|}^y. \end{aligned}$$

The first equality follows from Lemma 4.1(i), Lemma 4.19 and Lemma 4.24 and the assumptions, the third equality follows from $fq_{ji}^w h = 0$ for all $(j, i) \neq (1, |W_\lambda|)$ because $\deg(fq_{ji}^w h) > d_{y, \lambda}$ for all $(j, i) \neq (1, |W_\lambda|)$ by Lemma 4.24, the fourth equality follows from $q_{1, |W_\lambda|}^w$ is the identity map by Lemma 4.24 and the final equality follows from $\text{For}(\pi_{1, |W_\lambda|}^y) = \text{id}_{P_b^{y, \lambda}}$ and $\text{For}(\pi_{1, |W_\lambda|}^w) = \text{id}_{P_b^{w, \lambda}}$. Therefore $\tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y)$ and $\tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h))$ are equal.

Now we claim that $\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h) \in \mathbb{C}\theta_{y, 0}$. Otherwise, the head and the socle of the image of $\text{For}(\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h))$ are isomorphic to $L_b^{w, 0}$ such that they are not equal. We get a contradiction, by using Lemma 4.1(ii), to the result from computation above that the image of $\tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h))$ is a simple module. Hence $\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h) \in \mathbb{C}\theta_{y, 0}$. We also have $\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y \in \mathbb{C}\theta_{y, 0}$ because $\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y = \tilde{T}_\lambda^0(c\theta_{y, \lambda})\bar{\theta}_y = c\theta_{y, 0}$ by the assumption and (4.23) for some $c \in \mathbb{C}$. We have $\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y = \tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h)$ because they belong to $\mathbb{C}\theta_{y, 0}$ and $\tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y) = \tilde{T}_0^\lambda(\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h))$. This completes the proof of the first part of the proposition.

For the second part, we may assume $f \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w, \lambda}, \tilde{P}_b^{y, \lambda})_k$ for some $k \geq 0$ such that $f \neq 0$. There is a homomorphism $h \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{y, \lambda}, \tilde{P}_b^{w, \lambda})_{d_{y, \lambda} - j}$ such that $fh = \theta_{y, \lambda}$. Hence $\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y = \theta_{y, 0}$.

Since B_0^p is a symmetric algebra over \mathbb{C} [17, Theorem 4.6] and $\tilde{A}^p(0)$ is Koszul ([3, Theorem 1.1.3]), by Corollary 3.12 we know that $B_0^p = \text{End}_{\tilde{A}^p(0)}(\oplus_{w \in W_0} \tilde{P}_b^{w, 0})$ has an admissible basis and the canonical form tr attached to the basis is symmetrizing form. Hence

$$\begin{aligned} \text{tr}(\tilde{T}_\lambda^0(hf)\bar{\theta}_w) &= \text{tr}(\bar{\theta}_w\tilde{T}_\lambda^0(h)\tilde{T}_\lambda^0(f)) \\ &= \text{tr}(\tilde{T}_\lambda^0(f)\bar{\theta}_w\tilde{T}_\lambda^0(h)) \\ &= \text{tr}(\tilde{T}_\lambda^0(f)\tilde{T}_\lambda^0(h)\bar{\theta}_y) \\ &= \text{tr}(\theta_{y, 0}) \neq 0. \end{aligned}$$

Since $\tilde{T}_\lambda^0(hf)\bar{\theta}_w \in \text{End}_{\tilde{A}^p(0)}(\tilde{P}_b^{w \cdot 0})$ is homogeneous, it follows that $\tilde{T}_\lambda^0(hf)\bar{\theta}_w \in \mathbb{C}^\times \theta_{w \cdot 0}$ and $\deg(hf) = d_{w \cdot 0} - (d_{w \cdot 0} - d_{w \cdot \lambda}) = d_{w \cdot \lambda}$. Also $\deg(h) + \deg(f) = d_{y \cdot \lambda}$. This implies $d_{y \cdot \lambda} = d_{w \cdot \lambda}$. This completes the proof. \square

4.27. *Remark.* We conjecture that the element $\bar{\theta} = \sum_{w \in \check{W}^\lambda} \bar{\theta}_w$ commutes with the element $T_\lambda^0(h)$ for any $h \in \text{End}_{\tilde{A}^p(\lambda)}(\oplus_{w \in \check{W}^\lambda} \tilde{P}_b^{w \cdot \lambda})$.

Proof of Theorem 1.2: By Corollary 3.14, it is enough to show that $B_\lambda^p = \text{End}_{\tilde{A}^p(\lambda)}(\oplus_{w \in \check{W}^\lambda} \tilde{P}_b^{w \cdot \lambda})$ is a symmetric algebra. Recall that $B_0^p = \text{End}_{\tilde{A}^p(0)}(\oplus_{w \in \check{W}^0} \tilde{P}_b^{w \cdot 0})$.

Since B_0^p is a symmetric algebra over \mathbb{C} [17, Theorem 4.6] and $\tilde{A}^p(0)$ is Koszul ([3, Theorem 1.1.3]), by Corollary 3.12 we know that B_0^p has an admissible basis and the canonical form tr attached to the basis is symmetrizing form.

Recall that $\check{W}^\lambda \subset \check{W}^0$ by (4.9). Now we define a form tr_λ on B_λ^p by $\text{tr}_\lambda(g) = \text{tr}(T_\lambda^0(g)\bar{\theta})$ for all $g \in B_\lambda^p$. To show tr_λ is non-degenerate, it is enough to show that given $w, y \in \check{W}^\lambda$ and a nonzero homomorphism $g \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w \cdot \lambda}, \tilde{P}_b^{y \cdot \lambda})_j$ for some j , there is an $h \in B_\lambda^p$ such that $\text{tr}_\lambda(gh) \neq 0$.

By Lemma 3.5, there exists $h \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{y \cdot \lambda}, \tilde{P}_b^{w \cdot \lambda})_{d_{y \cdot \lambda} - j}$ such that $gh = \theta_{y \cdot \lambda}$. Therefore $\text{tr}_\lambda(gh) = \text{tr}_\lambda(\theta_{y \cdot \lambda}) = \text{tr}(\theta_{y \cdot \lambda} \bar{\theta}) = \text{tr}(\theta_{y \cdot 0}) = 1$. Hence tr_λ is non-degenerate. It remains to show that tr_λ is symmetric.

Let g be a nonzero homomorphism in $\text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{w \cdot \lambda}, \tilde{P}_b^{y \cdot \lambda})_j$ for $w, y \in \check{W}^\lambda$ and $j \in \mathbb{Z}$. By Proposition 4.14 or Proposition 4.25, we have $d_{w \cdot \lambda} = d_{y \cdot \lambda}$. It is clear that $\text{tr}_\lambda(gh) = \text{tr}_\lambda(hg) = 0$ for all $h \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{y' \cdot \lambda}, \tilde{P}_b^{w' \cdot \lambda})_k$ for $w', y' \in \check{W}^\lambda$ with $(w', y', k) \neq (w, y, d_{y \cdot \lambda} - j)$. For $h \in \text{Hom}_{\tilde{A}^p(\lambda)}(\tilde{P}_b^{y \cdot \lambda}, \tilde{P}_b^{w \cdot \lambda})_{d_{y \cdot \lambda} - j}$,

$$\begin{aligned} \text{tr}_\lambda(gh) &= \text{tr}(T_\lambda^0(gh)\bar{\theta}) = \text{tr}(T_\lambda^0(gh)\bar{\theta}_y) = \text{tr}(T_\lambda^0(g)T_\lambda^0(h)\bar{\theta}_y) \\ &= \text{tr}(T_\lambda^0(g)\bar{\theta}_w T_\lambda^0(h)) = \text{tr}(T_\lambda^0(h)T_\lambda^0(g)\bar{\theta}_w) = \text{tr}(T_\lambda^0(hg)\bar{\theta}_w) \\ &= \text{tr}_\lambda(hg), \end{aligned}$$

where we have used Proposition 4.25 for the fourth equality and the fifth equality follows from the fact that tr is a symmetrizing form. Therefore tr_λ is also a symmetrizing form. This completes the proof of Theorem 1.2.

4.28. **Corollary.** *For a dominant integral weight λ and P a projective-injective module in \mathcal{O}_λ^p , the endomorphism algebra $\text{End}_{\mathcal{O}_\lambda^p}(P)$ is a graded symmetric algebra which equips a homogeneous non-degenerate symmetric bilinear form of a degree $-d$, where d is the common graded length of all indecomposable projective-injective modules in \mathcal{O}_λ^p .*

Proof. The corollary follows from Theorem 1.2 and Corollary 3.14. \square

Finally, by the main results in [1], we know that the endomorphism algebra of any tilting module in \mathcal{O}_λ^p has a cellular structure. This can be generalized to the \mathbb{Z} -graded setting without difficulty. Note that every tilting module in \mathcal{O}_λ^p has a graded lift [16, Corollary 5].

4.29. **Proposition.** *(cf. [1]) Let λ be a dominant integral weight and Q a tilting module in \mathcal{O}_λ^p . Then the endomorphism algebra $\text{End}_{\mathcal{O}_\lambda^p}(Q)$ is a \mathbb{Z} -graded cellular algebra over \mathbb{C} in the sense of [7].*

In fact, when \mathfrak{g} is a semisimple Lie algebra of type A , Brundan and Kleshchev [4] show that the endomorphism algebra of the basic projective-injective module in the parabolic BGG category \mathcal{O}^p is essentially the basic algebra of the cyclotomic quiver Hecke algebra associated to the linear quiver (i.e., the degenerate cyclotomic Hecke algebras of type A). So Corollary 4.28 and Proposition 4.29 indicate that for any semisimple Lie algebra \mathfrak{g} , the endomorphism algebra of the basic projective-injective module in the parabolic BGG category \mathcal{O}^p behaves very much like the cyclotomic quiver Hecke algebra (cf. [4], [7], [21, Proposition 3.10]) and is a new class of interesting objects which deserves further study.

ACKNOWLEDGEMENTS

The first author was supported by the National Natural Science Foundation of China (No. 11525102, 11471315). The second author was partially supported by MoST grant 104-2115-M-006-015-MY3 of Taiwan.

REFERENCES

- [1] H. H. ANDERSEN, C. STROPPEL, D. TUBBENHAUER, *Cellular structures using U_q -tilting modules*, preprint, arXiv:1503.00224.
- [2] E. BACKELIN, *Koszul duality for parabolic and singular category \mathcal{O}* , Represent. Theory, **3** (1999), 139–152 (electronic).
- [3] A. BEILINSON, V. GINZBURG, AND W. SOERGEL, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc., **9** (1996), 473–527.
- [4] J. BRUNDAN AND A. KLESHCHEV, *Schur-Weyl duality for higher levels*, Selecta Math. (N.S.), **14** (2008), 1–57.
- [5] E. CLINE, B. PARSHALL, AND L. SCOTT, *The homological dual of a highest weight category*, Proc. London Math. Soc. (3), **68** (1994), 294–316.
- [6] R. GORDON AND E. L. GREEN, *Graded Artin algebras*, J. Algebra, **76** (1982), 111–137.
- [7] J. HU AND A. MATHAS, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math., **225** (2010), 598–642.
- [8] J. HU AND A. MATHAS, *Fayers’ conjecture and the socles of cyclotomic Weyl modules*, preprint, arXiv:1602.06631, 2016.
- [9] J. E. HUMPHREYS, *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* , Graduate Studies in Mathematics, **94**, American Mathematical Society, Providence, RI, 2008.
- [10] R. S. IRVING, *Projective modules in the category \mathcal{O}_S : self-duality*, Trans. Amer. Math. Soc., **291**(2) (1985), 701–732.
- [11] R. S. IRVING, *Projective modules in the category \mathcal{O}_S : Loewy series*, Trans. Amer. Math. Soc., **291**(2) (1985), 733–754.
- [12] R. S. IRVING AND B. SHELTON, *Loewy series and simple projective modules in the category \mathcal{O}_S* , Pacific Journal of Mathematics, **132**(2) (1988), 319–342.
- [13] J. C. JANTZEN, *Einhüllende Algebren halbeinfacher Lie-algebren*, Springer-Verlag, (1983).
- [14] R. MAKSIMAU, *Quiver Schur algebras and Koszul duality*, J. Algebra, **406** (2014), 91–133.
- [15] K. COULEMBIER AND V. MAZORCHUK, *Some homological properties of category \mathcal{O} . IV*, preprint, arXiv:1509.04391.
- [16] V. MAZORCHUK AND S. OVSIENKO, *A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra*, J. Math. Kyoto Univ., **45**(4) (2005), 711–741.
- [17] V. MAZORCHUK AND C. STROPPEL, *Projective-injective modules, Serre functors and symmetric algebras*, J. reine angew. Math., **616** (2008), 131–165.
- [18] A. ROCHA-CARIDI, *Splitting criteria for \mathfrak{g} -modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional, irreducible \mathfrak{g} -module*, Trans. Amer. Math. Soc., **262**(2) (1980), 335–366.
- [19] C. STROPPEL, *Category \mathcal{O} : gradings and translation functors*, J. Algebra, **268** (2003), 301–326.
- [20] C. STROPPEL, *Category \mathcal{O} : quivers and endomorphism rings of projectives*, Representation Theory, **7** (2003), 322–345.
- [21] P. SHAN, M. VARAGNOLO, E. VASSEROT, *On the center of quiver-Hecke algebras*, to appear, Duke Math. J., to appear, DOI 10.1215/00127094-3792705, (2017).

SCHOOL OF MATHEMATICS AND STATISTICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING, 100081, P.R. CHINA
E-mail address: junhu404@bit.edu.cn

DEPARTMENT OF MATHEMATICS, NATIONAL CHENG-KUNG UNIVERSITY, TAINAN, 70101, TAIWAN
E-mail address: nlam@mail.ncku.edu.tw